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Characterization, Parameter Estimation, and Aircraft Response Statistics of Atmospheric Turbulence

William D. Mark

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Characterization, Parameter Estimation, and Aircraft Response Statistics of Atmospheric Turbulence

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Prepared for
Langley Research Center
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LIST OF SYMBOLS

a_j	coefficient in parametric representation of ϕ_{w_s}
$a_n(t)$	coefficient in series expansions of ϕ_{w_f}
c_n	Fourier series coefficient of $w(t)$
E	least-squares error
$E\{\dots\}$	mathematical expectation operation
$E_z\{\dots\}$	mathematical expectation operation with respect to process $z(t)$
$E_{\sigma_f}\{\dots\}$	mathematical expectation operation with respect to process $\sigma_f(t)$
f	frequency
H	complex frequency response function
h	unit-impulse response function
\dot{h}	temporal derivative of $h(t)$
$\dot{H}(f)$	Fourier transform of $\dot{h}(t)$
H_k	Hermite polynomial of degree k
$\dot{h}(t)$	time derivative of $h(t)$
k	wavenumber
\bar{k}	dimensionless wavenumber
$k!$	k factorial
K_n	modified Bessel function of second kind of order n
L	integral scale, likelihood function
l	turbulence record length
\ln	natural logarithm

LIST OF SYMBOLS (Cont.)

L_z	integral scale of $z(t)$
m	degree of polynomial autocorrelation function representation
$m^{(k)}$	k th order power-moment spectrum
$m_n^{(k)}(f)$	k th order power-moment spectrum of $h(t)$
n	index
$\binom{n}{k}$	binomial coefficient
$N_+(y)$	expected up-crossing rate of level y
p	probability density function
P.D.F.	probability density function
$p^{(k)}(y \sigma_f)$	k th order derivative of $p(y \sigma_f)$ with respect to σ_y^2
P.S.D.	power spectral density
$p(S_n)$	probability density of S_n
$p(y)$	probability density of y
$p(y \sigma_f)$	conditional probability density of y given that $\sigma_f(u)$ is specified for all $-\infty < u \leq t$
$p(\sigma_f)$	probability density of σ_f
$p(\sigma_y^2)$	probability density of σ_y^2
R	empirical autocorrelation function
R_{N+1}	remainder term in series expansion of ϕ_{w_f}
\bar{S}	expected value of S
$S(f)$	estimate of power spectral density of $w(t)$
T	duration of turbulence record
t	time

LIST OF SYMBOLS (Cont.)

u	dummy time variable
V	aircraft speed
w	turbulence velocity
w_f	"fast" turbulence velocity component
w_s	"slow" turbulence velocity component
x	spatial coordinate
y	aircraft response
\dot{y}	temporal derivative of $y(t)$
y_j	response of system h_j to input x_j
y_k	response of system h_k to input x_k
y_s	aircraft response to "slow" turbulence component
z	Gaussian factor of "fast" turbulence component

LIST OF SYMBOLS (Cont.)

Greek Symbols:

Γ	gamma function
γ	reciprocal of relative variance of σ_y^2 , parameter in gamma probability density function
δ	Dirac delta function
η	normalized variable, $y/\sqrt{\sigma^2}$
θ	probability density function parameter
$\mu^{(k)}$	kth central moment
$\mu_{\sigma_y^2}^{(k)}$	kth central moment of σ_y^2
ν	frequency
ξ	lag variable, "dummy" variable
$\bar{\xi}$	dimensionless lag variable
π	3.1415926535...
ρ	correlation coefficient
σ	standard deviation
$\hat{\sigma}^2$	estimate of σ^2
σ_f	time-varying standard deviation of "fast" turbulence component
$\tilde{\sigma}_f$	infinite dimensional vector representation of $\sigma_f(t)$
$\sigma_{\dot{y}}$	standard deviation of $\dot{y}(t)$
$\overline{\sigma_y^2}$	mean value of σ_y^2
$\sigma_{\dot{y}}$	standard deviation of $\dot{y}(t)$
σ_{y_f}	standard deviation of $y_f(t)$
$\overline{\sigma_{y_f}^2}$	mean value of $\sigma_{y_f}^2$

LIST OF SYMBOLS (Cont.)

$\overline{\sigma_{y_f}^2}$	mean value of $\sigma_{y_f}^2$
σ_{y_s}	standard deviation of $y_s(t)$
τ	temporal lag variable
Φ	power spectral density
ϕ	autocorrelation function
ϕ_h	autocorrelation function of $h(t)$
$\phi_{\dot{h}}$	autocorrelation function of $\dot{h}(t)$
$\Phi_h(f, t)$	instantaneous power spectral density of $h(t)$
ϕ_{h^2}	autocorrelation function of $h^2(t)$
ϕ_K	transverse or longitudinal von Karman autocorrelation function
Φ_{KL}	longitudinal von Karman spectrum
Φ_{KT}	transverse von Karman spectrum
$\Phi^{(n)}$	nth derivative of Φ
Φ_w	power spectral density of $w(t)$
ϕ_w	autocorrelation function of $w(t)$
Φ_{w_f}	power spectral density of $w_f(t)$
ϕ_{w_f}	autocorrelation function of $w_f(t)$
$\Phi_w(f, t g_f)$	instantaneous spectral density of $w(t)$ given that $\sigma_f(t)$ is specified
Φ_{w_s}	power spectral density of $w_s(t)$

LIST OF SYMBOLS (*Cont.*)

ϕ_{w_s}	autocorrelation function of $w_s(t)$
$\tilde{\phi}_{w_s}$	parametric representation of ϕ_{w_s}
$\Phi_{x_j x_k}$	instantaneous cross-spectral density of processes $x_j(t)$ and $x_k(t)$
$\phi_{x_j x_k}$	instantaneous cross-correlation function of process $x_j(t)$ and $x_k(t)$
Φ_y	power spectral density of $y(t)$
ϕ_y	autocorrelation function of $y(t)$
$\phi_{\dot{y}}$	autocorrelation function of $\dot{y}(t)$
$\Phi_{y_j y_k}$	instantaneous cross-spectral density of processes $y_j(t)$ and $y_k(t)$
Φ_z	power spectral density of $z(t)$
ϕ_z	autocorrelation function of $z(t)$
$\phi(\xi)$	parametric representation of $\phi(\xi)$
ϕ_{σ_f}	autocorrelation function of $\sigma_f(t)$
$\Phi_{\sigma_f^2}$	power spectral density of $\sigma_f^2(t)$
$\phi_{\sigma_f^2}$	autocorrelation function of $\sigma_f^2(t)$

LIST OF SYMBOLS (*Cont.*)

Mathematical Symbols:

\triangleq	is defined as
\equiv	is identical to
(\cdot)	centered dot within parenthesis represents the location of a generic variable or parameter

SUMMARY

Several new methods are developed in this report for characterizing atmospheric turbulence, estimating the parameters of these characterizations using modern statistical methods, and computing relevant aircraft response statistics to such turbulence.*

In Section 1 of the report, a nongaussian model of atmospheric turbulence is postulated that accounts for readily observable features of turbulence velocity records, their autocorrelation functions, and their spectra. New methods for computing probability density functions and mean exceedance rates of a generic aircraft response variable are developed in Sections 2 and 6 respectively. In Section 3, a new method is developed for maximum likelihood estimation of the parameters of a spectrum of known functional form — i.e., the von Karman transverse and longitudinal spectral forms. Formulas for the variances of such estimates of the integral scale and intensity are derived in Section 5. The maximum likelihood method is combined with a least-squares approach to yield a method for estimating the autocorrelation function parameters of a two component model of turbulence in Section 4. Various related problems are treated in the Appendices. Implementation of the methods to turbulence velocity records is documented in another report.

*Sections 1 through 4 and Appendices A through E of this report were completed and submitted to NASA for review in May 1978, Section 5 was submitted to NASA in November 1978, and Section 6 and Appendices F through K were submitted in October 1980.

TURBULENCE MODEL

In early aeronautical work, turbulence records were described as gusts. For purposes of computing wing stresses etc., these gusts were modeled in the 1930's and 1940's as deterministic functions of time — e.g., Ref. 1.

During this same period of time, a great deal of fundamental work on the mathematical description of statistical phenomena was carried out. Basic mathematical theory of stationary time series was developed by Wiener [2,3], Khintchine [4], and Rice [5]. Wang and Uhlenbeck [6] provide an excellent description of the state of development in 1945. Useful aspects of this work were soon applied to engineering problems — e.g., James, Nichols, and Phillips [7]. The statistics book by Cramer [8] is a classic.

Also, during this same general period, significant progress was made by Taylor [9,10] and von Karman [11] in providing a statistical representation of turbulence. Autocorrelation functions and power spectra play a fundamental role in these representations. An early study of the power spectra of turbulence records was carried out by Clementson [12].

A principal reason why the power spectrum is so useful a description of random processes is that it possesses exact input-output relationships for linear time-invariant systems — the power spectrum of the output is the power spectrum of the input multiplied by the square of the magnitude of the system frequency response function. Lin [13] generally is given credit for being the first to compute the output (correlation function) of a mechanical system from a comparable description of its input. Liepmann [14] applied these ideas to the problem of buffeting.

Thus, by the early 1950's a methodology existed for computing response statistics of an aircraft from a statistical description of the turbulence excitation provided either by the autocorrelation function or its Fourier transform, the power spectral density. If the turbulence excitation is assumed to be stationary and Gaussian, and the aircraft is modeled as a linear time-invariant system, then the response process also is stationary and Gaussian. In this case, the power spectrum of the response provides a complete statistical description of the response process.

However, many turbulence records have a generally nonstationary appearance. This nonstationary appearance was

taken into account by Press *et al.* [15] by modeling turbulence records $w(t)$ as homogeneous Gaussian processes with slowly varying standard deviations — i.e.,

$$w(t) = \sigma(t) z(t) \quad , \quad (1.1)$$

where $z(t)$ is a stationary Gaussian process with zero mean value and unit variance, and $\sigma(t)$ is a nonnegative function of time that may be regarded as being either deterministic or stochastic. Considerable effort has gone into computation of the response statistics of aircraft using the above model. Much of this work was done by Press [15] and Houbolt [16]. Rice's famous formula [5] for the expected number of crossings of a stationary random function past a specified threshold plays a central role in these studies. Sidwell [17] has described Eq. (1.1) as the Press model of turbulence.

The studies of aircraft response statistics using the model of Eq. (1.1) implicitly assume (i) that fluctuations in $\sigma(t)$ occur slowly in comparison with those of $z(t)$, and (ii) that variations in $\sigma(t)$ are negligible over durations comparable with aircraft impulse-response-function durations. Conditions for the validity of assumption (i) are given by Mark and Fischer [18] and those for the validity of (ii) are given by Mark in [19]. When these conditions are satisfied, one requires only the power spectral density of $z(t)$ and the probability density function of $\sigma^2(t)$ to determine the level crossing rates and probability density function of an aircraft response variable of interest. The most pertinent information about the probability density of $\sigma^2(t)$ is contained in its mean value and second moment [19].

One cannot help but inquire how far we can progress toward computing aircraft response statistics by dropping the model of Eq. (1.1) and assuming only that the turbulence constitutes an arbitrary (generally nonGaussian) stationary random process. This question was addressed by Mazelsky [20] in 1954 and slightly earlier in the Russian literature — in a more general context — by Kuznetsov and his associates [21]. Mazelsky and Kuznetsov showed that higher-order autocorrelation functions which are time-averaged lagged products obtained by multiplying turbulence records by themselves three, four, and more times also possess exact input-output relations — as do their various multidimensional Fourier transforms. However, computation times, problems of statistical confidence of estimates of these characterizations obtained from turbulence records of finite duration, and

problems of interpretation collectively make these higher-order correlation functions and spectra much less attractive for the characterization of measured records than the conventional autocorrelation function and power spectral density.

Our approach, therefore, is to postulate a turbulence model sufficiently general to include all readily observable features of measured turbulence records important to aircraft responses, and to develop turbulence characterizations from this model that have input-output relations sufficiently general to predict output probability density functions and threshold crossing rates for arbitrary aircraft response variables. A requirement of these turbulence characterizations absolutely essential for practical application is that it be possible to generate realizations of these characterizations from measured turbulence records.

Three Component NonGaussian Turbulence Model

The simplest model of turbulence as a stochastic process is that of a stationary Gaussian process. The vertical record shown in Fig. 1 (Ref. 22) illustrates a turbulence record that would appear to be reasonably well modeled as a stationary Gaussian time history — especially the portion of the record from 120 to 270 sec elapsed time. We might reasonably model this record using Eq. (1.1) with $\sigma(t)$ taken to be a constant. However, most records of atmospheric turbulence have a general appearance that is closer to that shown in Fig. 2 (Ref. 22). Notice, for example, that the portion of the vertical record between 135 and 145 sec elapsed time has a relatively small rms value; whereas, patches with much larger rms values occur shortly thereafter in the neighborhoods of 150 and 160 sec elapsed time. Such behavior cannot be modeled by a stationary Gaussian process, but can be reasonably modeled by Eq. (1.1) when $\sigma(t)$ is allowed to depend on time.

Each of the records shown in Fig. 2 also exhibits an additive weak low-frequency component that appears to fluctuate independently of the occurrence of the patches. For example, during the 5-sec interval between 183 and 188 sec elapsed time on the vertical record, high-frequency fluctuations are absent; however, there remains in that interval a fluctuating weak low-frequency component. Similar but more pronounced behavior of this type occurs between approximately

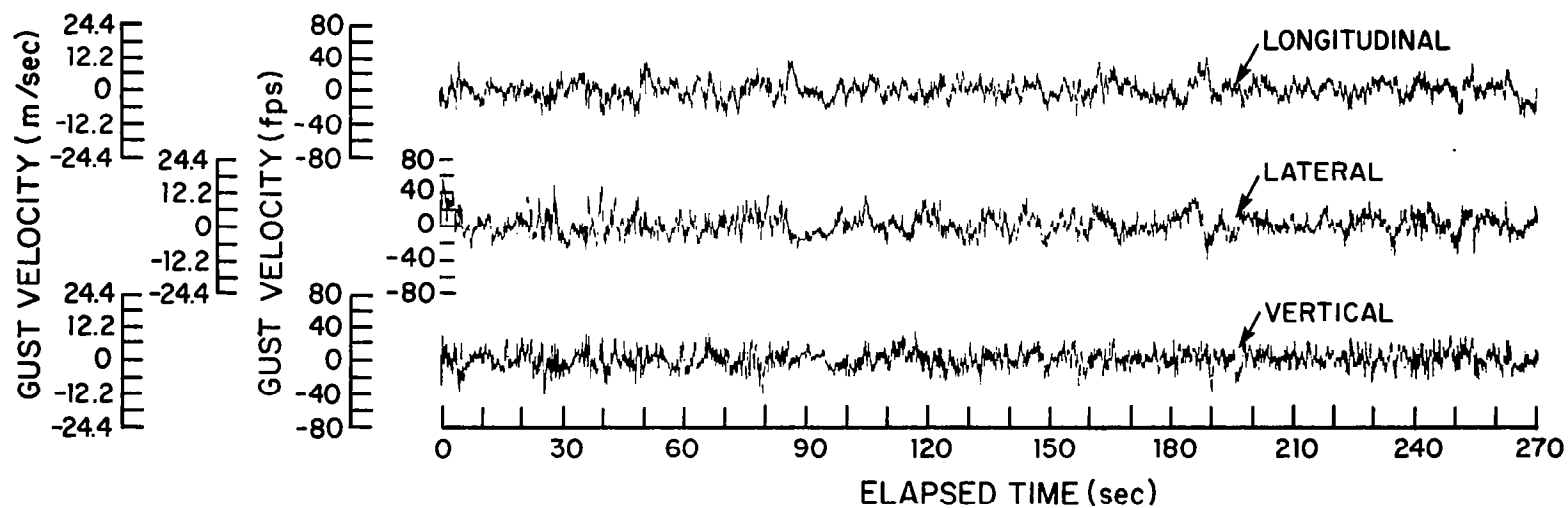


FIG. 1. TURBULENCE RECORDS WITH REASONABLY STATIONARY GAUSSIAN BEHAVIOR.
(Ref. 22, Fig. 17.32. p. 223.)

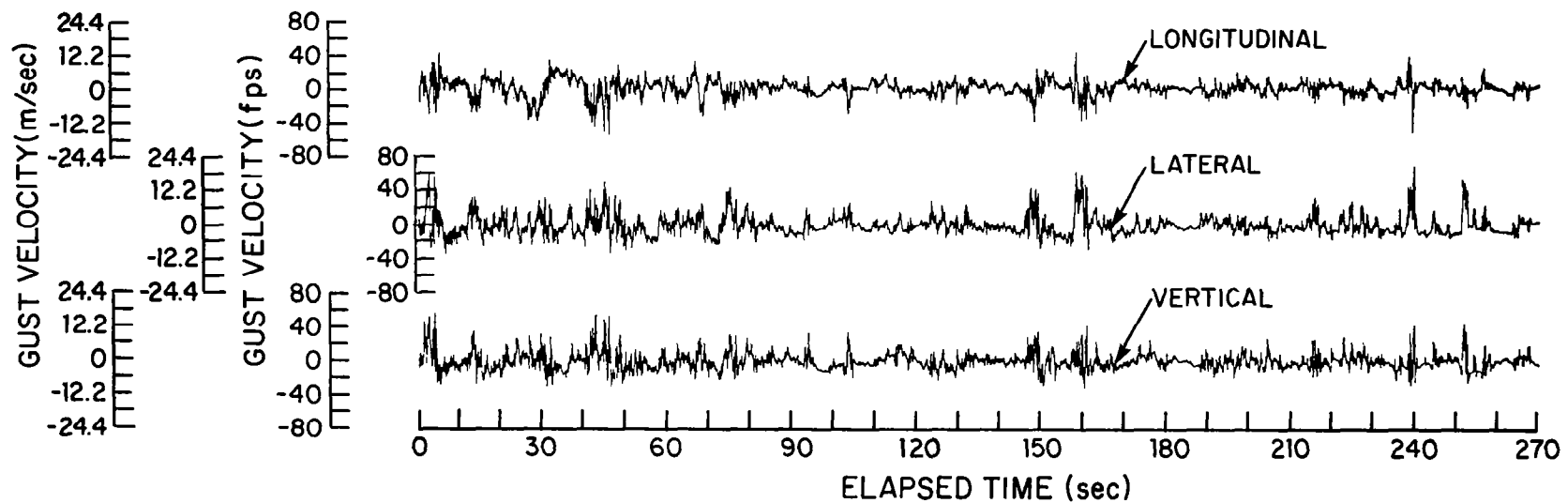


FIG. 2. TURBULENCE RECORDS WITH MILD PATCH-LIKE NONGAUSSIAN BEHAVIOR.
(Ref. 22, Fig. 17.28, p. 219.)

96 and 99 sec elapsed time on the vertical record shown in Fig. 3 (Ref. 22). High-frequency fluctuations are absent in that interval; however, there exists there a relatively strong low-frequency component. Such behavior cannot be modeled by Eq. (1.1). Many other excellent records showing similar behavior may be found in Ref. 22.

The above discussion suggests that we add a low-frequency component to the model of Eq. (1.1) - i.e., that we postulate a turbulence model [19] of the form

$$w(t) = w_s(t) + w_f(t) \quad (1.2a)$$

$$= w_s(t) + \sigma_f(t) z(t) \quad , \quad (1.2b)$$

where

$$w_f(t) = \sigma_f(t) z(t) \quad , \quad \sigma_f(t) \geq 0 \quad , \quad (1.3)$$

and

$$E\{z(t)\} = 0 \quad , \quad E\{z^2(t)\} = 1 \quad . \quad (1.4)$$

In Eq. (1.2a), $w_s(t)$ is the "slow" (low-frequency) component and $w_f(t)$ is the "fast" intensity modulated component described by Eq. (1.3). Since well-behaved turbulence records such as the portion of the vertical record shown in Fig. 1 between 120 to 270 sec elapsed time generally have Gaussian (first-order) probability density functions, we shall further assume that the stochastic process $z(t)$ in the above model is stationary and Gaussian. In some of the work to follow, we also shall assume that $w_s(t)$ is a stationary Gaussian process. Thus, the processes $\{w_s(t)\}$ and $\{z(t)\}$ are fully described by their power spectral densities or autocorrelation functions. We also shall generally assume that $\sigma_f(t)$ is a stationary random process; however, since $\sigma_f(t) > 0$, we shall not assume that $\sigma_f(t)$ is Gaussian. Furthermore, the three processes $\{w_s(t)\}$, $\{\sigma_f(t)\}$, and $\{z(t)\}$ will be assumed to be statistically independent.

Each of the turbulence records shown in Fig. 4 (Ref. 23) clearly illustrates the three turbulence components $w_s(t)$, $\sigma_f(t)$, and $z(t)$ in the model of Eq. (1.2). Notice that throughout each entire record a strong low-frequency component $w_s(t)$ is present. However, for example, between 9 min 0 sec and 9 min 45 sec on each record, $w_f(t)$ is negligible in

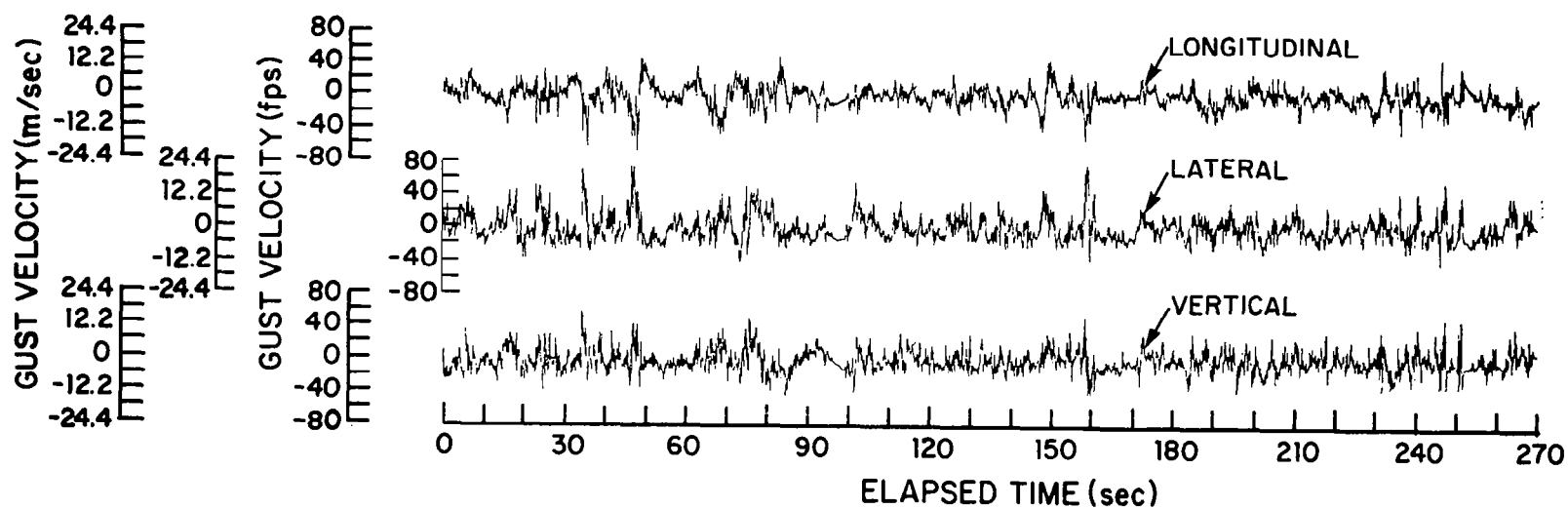


FIG. 3. TURBULENCE RECORDS WITH ADDITIVE LOW-FREQUENCY COMPONENTS PRESENT.
(Ref. 22, Fig. 17.34, p. 225.)

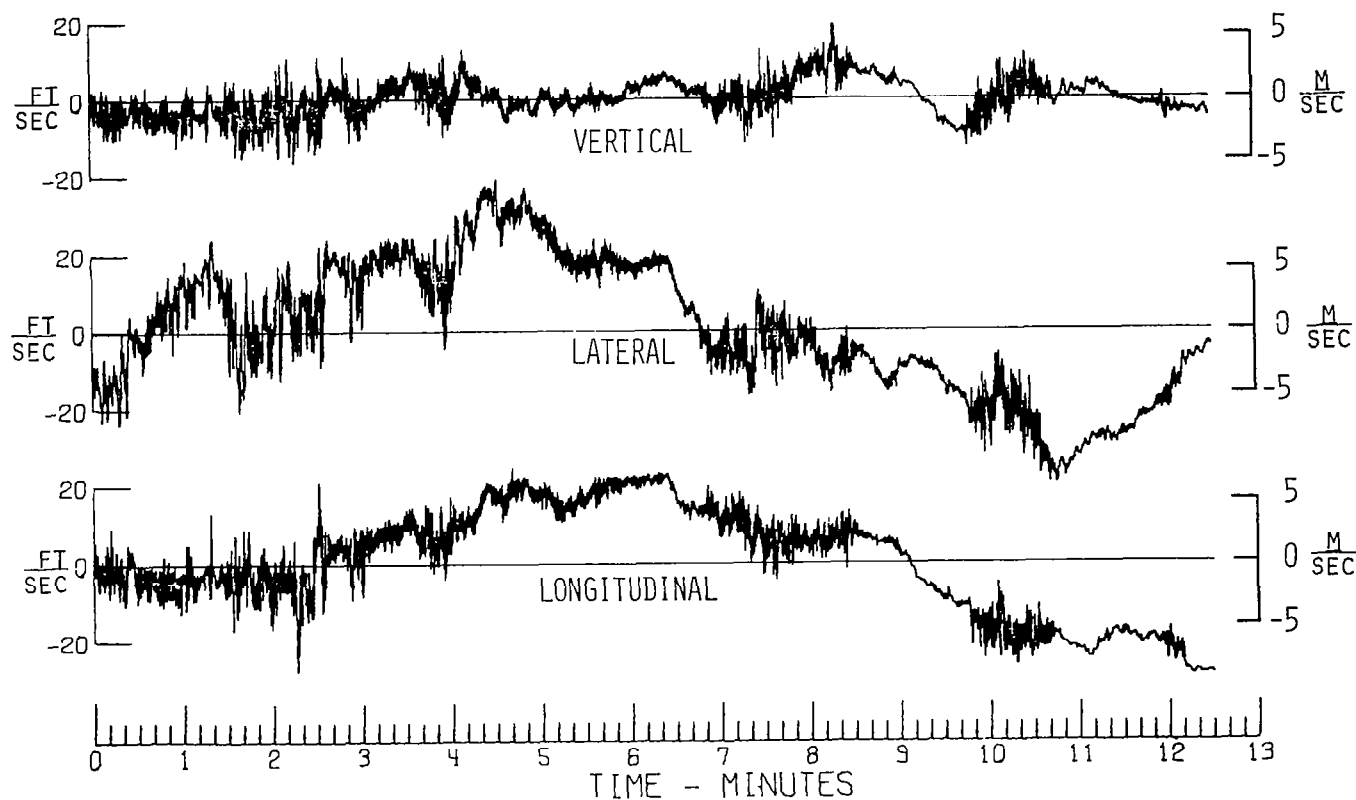


FIG. 4. TURBULENCE RECORDS CONTAINING THE INDIVIDUAL COMPONENTS $w_s(t)$, $\sigma_f(t)$, AND $z(t)$ IN THE THREE COMPONENT MODEL OF EQ. (1.2). [MOUNTAIN WAVE CONDITIONS. AIRCRAFT SPEED 197 m/sec (646 ft/sec).] (Ref. 23, Fig. 10, p. 285.)

comparison with $w_s(t)$; whereas, between about 9 min 45 sec and 10 min 45 sec, $w_f(t)$ grows and then decays back to a small value again. Such behavior is controlled by the temporal variations in $\sigma_f(t)$. None of the three records shown in Fig. 4 could satisfactorily be modeled by a single stationary Gaussian process — or by the process model of Eq. (1.1). *Equation (1.2) is the simplest model capable of describing the overall turbulence behavior illustrated in Fig. 4.*

Locally stationary assumption. For much of the work to follow, we shall further assume that fluctuations in $\sigma_f(t)$ occur slowly in comparison with those of $z(t)$. The quantitative statement of this assumption [18,19] must be made in terms of the power spectral density of the process $\{z(t)\}$. Let $\Phi_z(f)$ be the two-sided power spectral density of $\{z(t)\}$, which we define here as the Fourier transform of the autocorrelation function of $\{z(t)\}$ — i.e.,

$$\Phi_z(f) \triangleq \int_{-\infty}^{\infty} \phi_z(\tau) e^{-i2\pi f\tau} d\tau, \quad (1.5)$$

where

$$\phi_z(\tau) \triangleq E\{z(t) z(t+\tau)\}. \quad (1.6)$$

Also let $\Phi_z^{(2)}(f)$ be the second derivative of $\Phi_z(f)$:

$$\Phi_z^{(2)}(f) \triangleq \frac{d^2}{df^2} \Phi_z(f). \quad (1.7)$$

Then our locally stationary assumption may be expressed as

$$\left| \frac{d^2 \ln \sigma_f^2(t)}{dt^2} \right| \ll 32\pi^2 \frac{\Phi_z(f)}{|\Phi_z^{(2)}(f)|}. \quad (1.8a)$$

The requirement of Eq. (1.8a) is derived and discussed in Ref. 18 and is further discussed in Ref. 19. See, in particular, Eq. (4.53) and pp. 43 to 53 of Ref. 18. When $\Phi_z(f)$ has the form of the von Karman transverse spectrum, with integral scale L_z , it is shown in p. 51 of Ref. 19 that the requirement of Eq. (1.8a) reduces to

$$\left| \frac{d^2 \ln \sigma_f^2(t)}{dt^2} \right| \leq 0.08 \frac{V^2}{L_z^2}, \quad (1.8b)$$

where V is the aircraft speed. This requirement is more meaningfully expressed in terms of σ_f written as a function of the spatial variable $x = Vt$:

$$\left| \frac{d^2 \ln \sigma_f^2(x/V)}{dx^2} \right| \leq \frac{0.08}{L_z^2}. \quad (1.8c)$$

The examples of this requirement discussed on pp. 43 to 50 of Ref. 18 show that when a typical length scale L_G of the function $\sigma_f(x/V)$ is at least 10 times L_z , the locally stationary assumption of Eq. (1.8) is well satisfied.

A more relaxed locally stationary condition expressed in terms of the autocorrelation function of $\ln \sigma_f^2(t)$ and the aircraft frequency response function of interest is given by Eq. (5.17) on p. 56 of Ref. 19. An expression for the autocorrelation function of $\ln \sigma_f^2(t)$ in terms of measurable quantities is given by Eq. (5.36) on p. 61 of Ref. 19.

For typical atmospheric turbulence records, the locally stationary assumption of Eq. (1.8) is believed to introduce negligible error in the results to follow. The simulation studies carried out in Ref. 24 support this conclusion.

Spectral form of $z(t)$. In some of the work to follow, we shall assume that the process $\{z(t)\}$ has the appropriate (transverse or longitudinal) von Karman spectral form. When the locally stationary condition of Eq. (1.8) is satisfied, it is shown in Ref. 18 that the spectral form of $w_f(t) = \sigma_f(t) \times z(t)$ is unaffected by the fluctuations in $\sigma_f(t)$ — i.e., $w_f(t)$ will have the same form of spectrum as $\{z(t)\}$.

Comparison with previous related models. A three-component turbulence model functionally similar to Eq. (1.2) has been studied by Reeves *et al.* [25–27] and Sidwell [28]. However, both Reeves and Sidwell assume that their counterpart to our $\sigma_f(t)$ is a Gaussian process with zero mean value.

This restriction forces their counterpart to $w_f(t)$ to have periods of very low intensity [when $\sigma_f(t)$ is near its mean value of zero]. Reeves motivation [25-27] for adding his counterpart to $w_s(t)$ is to partially remove such "deep fades." Thus, in Reeves' model, the power spectra of his counterparts to $w_s(t)$ and $w_f(t)$ are taken to have the same Dryden form.

On the other hand, our approach in Refs. 18 and 19, and in the present work, is to introduce a minimum of assumptions pertaining to the behavior of $\sigma_f(t)$ and $w_s(t)$ and to extract descriptions of these processes relevant to the aircraft response problems from measured turbulence records. Algorithms for computing the power spectra of $z(t)$, $w_s(t)$, and $\sigma_f^2(t)$, and for computing moments and probability density functions of $w_s(t)$ and $\sigma_f^2(t)$ are provided in Sec. 6 of Ref. 19. Some of these techniques are modified and extended in the present work.

Aircraft Response Metrics

Aircraft model. It is our goal to develop expressions for the (first-order) probability density functions and the threshold mean crossing rates for a general aircraft response variable — using the above described model of turbulence as the excitation or input. We shall model the aircraft as a linear two-terminal time-invariant system described either by its unit-impulse response function $h(t)$ or complex frequency response $H(f)$:

$$H(f) \triangleq \int_{-\infty}^{\infty} h(t) e^{-i2\pi ft} dt \quad . \quad (1.9)$$

For any turbulence sample function $w(t)$, the aircraft response $y(t)$ is the convolution of $w(t)$ and $h(t)$ — i.e.,

$$y(t) = \int_{-\infty}^{\infty} h(\tau) w(t-\tau) d\tau \quad (1.10a)$$

$$= \int_{-\infty}^{\infty} w(\tau) h(t-\tau) d\tau \quad . \quad (1.10b)$$

Implicit in this treatment is the assumption that the spatial variation of $w(t)$ is negligible over dimensions comparable to those of the aircraft. The aircraft impulse response may represent the response of any aircraft variable of interest — e.g., the stress at a critical point in a wing span — to the turbulence velocity input $w(t)$. The impulse response $h(t)$ may include the action of a pilot or autopilot modeled as a linear feedback element. That is, $h(t)$ may be thought of as either an open or closed loop unit-impulse response function.

Autocorrelation function input-output relationship. It is well known — e.g., p. 71 of Ref. 29 — that the power spectral density $\Phi_y(f)$ of the aircraft response $y(t)$ is related to the input power spectrum $\Phi_w(f)$ and aircraft frequency response $H(f)$ by

$$\Phi_y(f) = \Phi_w(f) |H(f)|^2 . \quad (1.11)$$

Let $\phi_w(\tau)$ and $\phi_y(\tau)$ denote, respectively, the autocorrelation functions of the excitation and response processes $\{w(t)\}$ and $\{y(t)\}$:

$$\phi_w(\tau) = \int_{-\infty}^{\infty} \Phi_w(f) e^{i2\pi f\tau} df \quad (1.12)$$

$$\phi_y(\tau) = \int_{-\infty}^{\infty} \Phi_y(f) e^{i2\pi f\tau} df . \quad (1.13)$$

Then, from the convolution theorem and Eq. (1.11), it follows that the autocorrelation function input-response relationship is the convolution

$$\phi_y(\tau) = \int_{-\infty}^{\infty} \phi_h(\tau-\xi) \phi_w(\xi) d\xi \quad (1.14a)$$

$$= \int_{-\infty}^{\infty} \phi_h(\xi-\tau) \phi_w(\xi) d\xi \quad (1.14b)$$

$$= \int_{-\infty}^{\infty} \phi_h(\xi+\tau) \phi_w(\xi) d\xi , \quad (1.14c)$$

where

$$\phi_h(\tau) \triangleq \int_{-\infty}^{\infty} |H(f)|^2 e^{i2\pi f\tau} df, \quad (1.15a)$$

$$= \int_{-\infty}^{\infty} h(t) h(t+\tau) dt, \quad (1.15b)$$

and where Eq. (1.14b) follows from Eq. (1.14a) and the fact that the autocorrelation function $\phi_h(\tau)$ is an even function of τ , and Eq. (1.14c) follows from the fact that $\phi_y(\tau)$ is an even function of τ .

Since we have assumed that the turbulence components $\{w_s(t)\}$, $\{\sigma_f(t)\}$, and $\{z(t)\}$ are mutually statistically independent, it follows that $\{w_s(t)\}$ and $\{w_f(t)\}$ also are mutually independent (and therefore uncorrelated). Therefore, from Eq. (1.2) it follows that

$$\phi_w(\tau) = \phi_{w_s}(\tau) + \phi_{w_f}(\tau) \quad (1.16a)$$

$$= \phi_{w_s}(\tau) + \phi_{\sigma_f}(\tau) \phi_z(\tau), \quad (1.16b)$$

where the second line is a consequence of Eq. (1.3) and the assumed independence of $\{\sigma_f(t)\}$ and $\{z(t)\}$. Moreover, from the locally stationary assumption of Eq. (1.8) it follows [18] that

$$\phi_{w_f}(\tau) \equiv \phi_{\sigma_f}(\tau) \phi_z(\tau) \quad (1.17a)$$

$$\approx \phi_{\sigma_f}(0) \phi_z(\tau) \quad (1.17b)$$

$$= E\{\sigma_f^2\} \phi_z(\tau); \quad (1.17c)$$

hence, we have from Eqs. (1.16c) and (1.17c)

$$\phi_w(\tau) \approx \phi_{w_s}(\tau) + E\{\sigma_f^2\} \phi_z(\tau). \quad (1.18)$$

That is, our turbulence model of Sec. 1.1 implies that the autocorrelation function of the turbulence $\{w(t)\}$ will appear as a superposition of the autocorrelation function of the slow component $w_s(t)$ and an amplitude scaled version of the autocorrelation $\phi_z(t)$ of the Gaussian component $z(t)$.

The behavior of our turbulence model described by Eq. (1.18) is illustrated in Fig. 5 for idealized autocorrelation functions. Notice that $\phi_{w_s}(\tau)$ is shown decaying much more slowly than $\phi_z(\tau)$. This latter behavior is consistent with our assumption that the process $w_s(t)$ fluctuates slowly in comparison with the process $z(t)$ — as is evident from the record shown in Fig. 4. In fact, $w_s(t)$ may be thought of as a slowly varying mean wind; whereas, $w_f(t) = \sigma_f(t) z(t)$ may be thought of as ordinary turbulence.

Substituting Eqs. (1.16 to 1.18) into Eq. (1.14c), we may express the aircraft response autocorrelation function as

$$\phi_y(\tau) = \int_{-\infty}^{\infty} \phi_h(\xi+\tau) [\phi_{w_s}(\xi) + \phi_{w_f}(\xi)] d\xi \quad (1.19a)$$

$$= \int_{-\infty}^{\infty} \phi_h(\xi+\tau) [\phi_{w_s}(\xi) + \phi_{\sigma_f}(\xi) \phi_z(\xi)] d\xi \quad (1.19b)$$

$$\approx \int_{-\infty}^{\infty} \phi_h(\xi+\tau) [\phi_{w_s}(\xi) + E\{\sigma_f^2\} \phi_z(\xi)] d\xi \quad (1.19c)$$

which are the desired autocorrelation function input-response relationships. Equations (1.19a) and (1.19b) are an exact consequence of the turbulence model described by Eq. (1.2) and the assumed independence of $\{w_s(t)\}$, $\{\sigma_f(t)\}$, and $\{z(t)\}$; whereas, in Eq. (1.19c), the locally stationary assumption of Eq. (1.8) has been used.

Aircraft mean-square displacement and velocity responses. Setting $\tau = 0$ in Eq. (1.19) directly yields the mean-square aircraft displacement response in terms of the turbulence component autocorrelation functions:

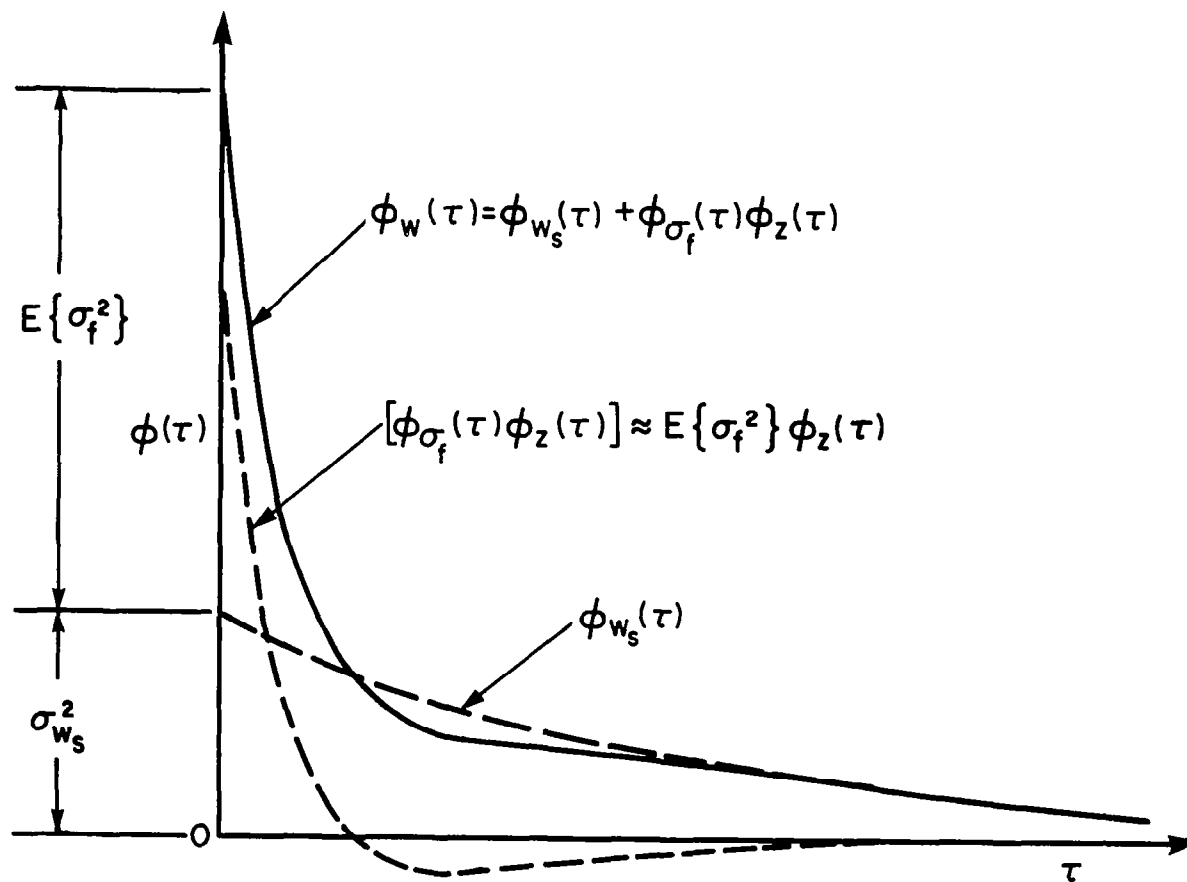


FIG. 5. IDEALIZED SKETCH OF AUTOCORRELATION FUNCTION OF ATMOSPHERIC TURBULENCE AND AUTOCORRELATION FUNCTION OF ITS COMPONENTS.

$$\begin{aligned}
E\{y^2\} &= \phi_y(0) \\
&= \int_{-\infty}^{\infty} \phi_h(\xi) [\phi_{w_s}(\xi) + \phi_{w_f}(\xi)] d\xi \quad (1.20a)
\end{aligned}$$

$$= \int_{-\infty}^{\infty} \phi_h(\xi) [\phi_{w_s}(\xi) + \phi_{\sigma_f}(\xi) \phi_z(\xi)] d\xi \quad (1.20b)$$

$$\approx \int_{-\infty}^{\infty} \phi_h(\xi) [\phi_{w_s}(\xi) + E\{\sigma_f^2\} \phi_z(\xi)] d\xi, \quad (1.20c)$$

where the approximation of Eq. (1.20c) again depends on the locally stationary assumption of Eq. (1.8).

Comparable expressions may be written for the aircraft mean-square velocity response $E\{\dot{y}^2\}$. If we formally differentiate Eq. (1.10), we obtain

$$\dot{y}(t) = \int_{-\infty}^{\infty} w(\tau) \dot{h}(t-\tau) d\tau, \quad (1.21)$$

where $\dot{h}(t)$ is the time derivative of the displacement impulse response $h(t)$. Since some $h(t)$ of interest may contain discontinuities — e.g., at $t = 0$ — care must be taken in computing $\dot{h}(t)$. That is, $\dot{h}(t)$ must satisfy

$$h(t) = \int_{-\infty}^t \dot{h}(\xi) d\xi. \quad (1.22)$$

Thus, if $h(t)$ has a discontinuity then $\dot{h}(t)$ must contain a delta function at the same place so that Eq. (1.22) is satisfied.

As in Eq. (1.15b), we may define for $h(t)$

$$\phi_{\dot{h}}(\xi) \triangleq \int_{-\infty}^{\infty} \dot{h}(t) \dot{h}(t+\tau) dt. \quad (1.23)$$

The mean-square velocity response may then be expressed in the same manner as in Eq. (1.20):

$$\begin{aligned} E\{\dot{y}^2\} &\equiv \phi_{\dot{y}}(0) \\ &= \int_{-\infty}^{\infty} \phi_{\dot{h}}(\xi) [\phi_{w_s}(\xi) + \phi_{w_h}(\xi)] d\xi \end{aligned} \quad (1.24a)$$

$$= \int_{-\infty}^{\infty} \phi_{\dot{h}}(\xi) [\phi_{w_s}(\xi) + \phi_{\sigma_f}(\xi) \phi_z(\xi)] d\xi \quad (1.24b)$$

$$\approx \int_{-\infty}^{\infty} \phi_{\dot{h}}(\xi) [\phi_{w_s}(\xi) + E\{\sigma_f^2\} \phi_z(\xi)] d\xi, \quad (1.24c)$$

where the approximation of Eq. (1.24c) again depends on the locally stationary assumption of Eq. (1.8).

Parametric Description of Atmospheric Turbulence

In order to tabulate relevant features of measured turbulence records, it is necessary to characterize such records by a set of parameters that can be extracted from the records. The integral scale and mean-square value of a record are examples of such parameters.

Characterization of "slow" component $w_s(t)$. The NASA MAT program (Measurement of Atmospheric Turbulence) [30] has concentrated on obtaining atmospheric turbulence recordings accurate to frequencies (wavenumbers) well below typical positions of the "knees" of von Karman spectra. This effort has required exceptional care in aircraft instrumentation [31]. Typical autocorrelation functions computed from MAT records suggest that an efficient characterization of autocorrelation function $\phi_{w_s}(\xi)$ of the low-frequency component $w_s(t)$ is a low-order polynomial approximation to $\phi_{w_s}(\xi)$ valid in the neighborhood of $\xi = 0$ [pp. 64,65 of Ref. 19]. Although this low-order polynomial representation may be interpreted as the first few terms of a Maclaurin expansion of $\phi_{w_s}(\xi)$, we shall use a (constrained) least-squares procedure to compute the actual expansion coefficients as described in Sec. 4 of this report.

Figure 6 displays the autocorrelation function of the vertical record shown in Fig. 4. According to Fig. 6, about 78% of the "power" in the vertical record of Fig. 4 is in $w_s(t)$ and about 22% is in $w_f(t)$.

To motivate our concentration on $\phi_w(\xi)$ in the region of ξ near the origin, we note first that to predict the first-order probability density functions and threshold mean crossing rates of aircraft responses, we require $E\{y^2\}$ and $E\{\dot{y}^2\}$ [pp. 34-50 of Ref. 19]. Expressions for these quantities are given by Eqs. (1.20) and (1.24) of this report. If we decompose $E\{y^2\}$ and $E\{\dot{y}^2\}$ into contributions from our "slow" and "fast" components $w_s(t)$ and $w_f(t)$, respectively,

$$E\{y^2\} = E\{y_s^2\} + E\{y_f^2\} \quad (1.25)$$

and

$$E\{\dot{y}^2\} = E\{\dot{y}_s^2\} + E\{\dot{y}_f^2\} , \quad (1.26)$$

then we see from Eqs. (1.20) and (1.24) that the slow component contributions are given by

$$E\{y_s^2\} = \int_{-\infty}^{\infty} \phi_h(\xi) \phi_{w_s}(\xi) d\xi \quad (1.27)$$

and

$$E\{\dot{y}_s^2\} = \int_{-\infty}^{\infty} \dot{\phi}_h(\xi) \phi_{w_s}(\xi) d\xi . \quad (1.28)$$

Let us now assume — for the purposes of the present discussion — that $h(t)$ and $\dot{h}(t)$ are of duration T_h sec only — e.g., that $h(t)$ and $\dot{h}(t)$ are zero outside the interval $0 < t < T_h$. Then, it is easy to show from Eqs. (1.15b) and (1.23), respectively that $\phi_h(\xi)$ and $\dot{\phi}_h(\xi)$ are zero for $|\xi| > T_h$. Consequently, from Eqs. (1.27) and (1.28), we see that $E\{y_s^2\}$ and $E\{\dot{y}_s^2\}$ depend on the values of $\phi_{w_s}(\xi)$ only for values of ξ satisfying $|\xi| \leq T_h$.

VERTICAL GUST VELOCITY AUTOCORRELATION

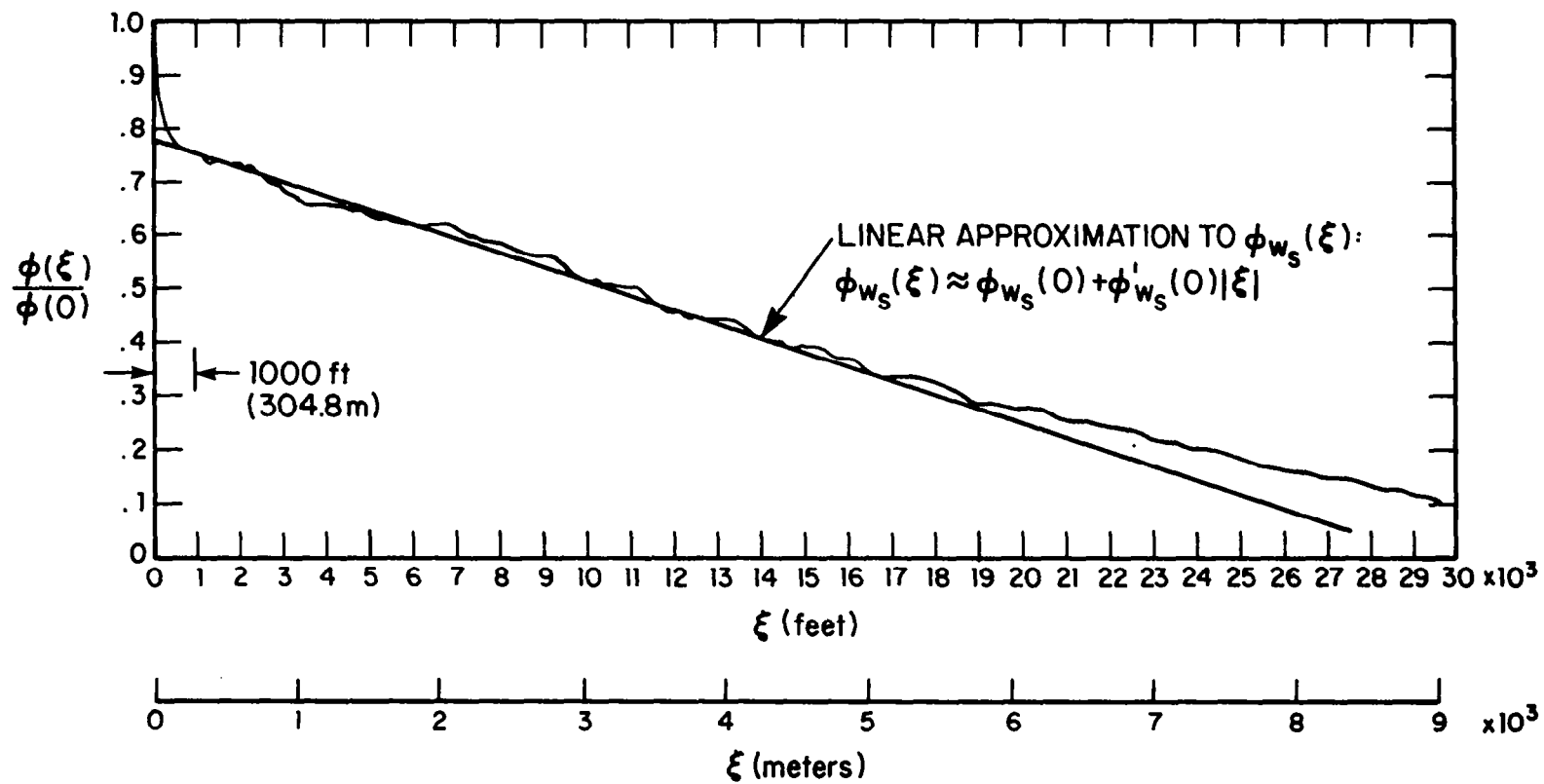


FIG. 6. AUTOCORRELATION FUNCTION OF VERTICAL RECORD SHOWN IN FIG. 4. [MOUNTAIN WAVE CONDITIONS. AIRCRAFT SPEED 197 m/sec (646 ft/sec.)]

As an example [23], we consider an aircraft flying at Mach 2.7 at an altitude of 18 km (60,000 ft) where the speed of sound is approximately 297 m/sec (975 ft/sec). If we take $T_h = 3$ sec, then the correlation interval of interest is about 2400 m (7900 ft). On the other hand, if we take $T_h = 10$ sec, the correlation interval of interest is about 8000 m (26,200 ft). For the autocorrelation function displayed in Fig. 6, we see that, for the 3-sec impulse response, a linear approximation to ϕ_{w_s} provides a good fit; whereas, for the 10-sec impulse response, a quadratic approximation would be adequate. Consequently, we shall take for our parametric representation $\phi_{w_s}(\xi)$ of $\phi_{w_s}(\xi)$:

$$\phi_{w_s}(\xi) \triangleq \sum_{j=0}^m a_j \xi^j, \quad (1.29)$$

where the degree m chosen for the above polynomial representation of $\phi_{w_s}(\xi)$ is to depend on the observed complexity of $\phi_{w_s}(\xi)$ and the interval in ξ over which $\phi_{w_s}(\xi)$ is to be represented by Eq. (1.29).

Two fundamentally different approaches may be used in generating the representation of Eq. (1.29). On the one hand, we may take the expansion interval as $0 \leq \xi \leq T_h$ and include both odd and even powers of ξ in the expansion. This procedure clearly is the best for the autocorrelation function shown in Fig. 6. On the other hand, we may take for the expansion interval, the even interval $-T_h \leq \xi \leq T_h$. Since, by definition, $\phi_{w_s}(\xi)$ must be an even function of ξ , this latter approach must contain only *even* powers of ξ . Generally, the latter approach will require higher powers of ξ to get a good fit using Eq. (1.29) — i.e., a larger value of m — but it has the advantage that the integrals obtained by substituting Eq. (1.29) into Eqs. (1.27) and (1.28) are, in some situations, more easily evaluated.

The reader may wish to interpret Eq. (1.29) as a truncated Maclaurin series expansion. In the first of the above two approaches where Eq. (1.29) applies to the interval $0 \leq \xi \leq T_h$, the derivatives of $\phi_{w_s}(\xi)$ in the interpretation must be considered as one-sided derivatives valid only in the region $\xi \geq 0$ — i.e., we have $a_j = \phi^{(j)}(0+)/j!$, where $\phi^{(j)}(0+)$

denotes the j th order derivative evaluated at $\xi = 0+$. In the second of the above two approaches where Eq. (1.29) applies to the interval $-T_h \leq \xi \leq T_h$, we have $a_j = \phi^{(j)}(0)/j!$, where here no "one-sided condition" is required since, in this interpretation, it is assumed that $\phi^{(j)}(\xi)$ is continuous at $\xi = 0$ for $j = 0, 1, 2, \dots, m$.

Since the integrands of Eqs. (1.27) and (1.28) are necessarily even, our estimates of $E\{y_s^2\}$ and $E\{\dot{y}_s^2\}$ obtained using Eq. (1.29) may be expressed as

$$E\{y_s^2\} \approx 2 \sum_{j=0}^m a_j \int_0^\infty \xi^j \phi_h(\xi) d\xi \quad (1.30)$$

and

$$E\{\dot{y}_s^2\} \approx 2 \sum_{j=0}^m a_j \int_0^\infty \xi^j \phi_h(\xi) d\xi. \quad (1.31)$$

Equations (1.30) and (1.31) are valid for either of the above types of expansion. However, from the Fourier mate to Eq. (1.15a), we have

$$|H(f)|^2 = \int_{-\infty}^\infty \phi_h(\xi) e^{-i2\pi f\xi} d\xi. \quad (1.32)$$

Differentiating Eq. (1.32) j times, we find

$$\frac{d^j |H(f)|^2}{df^j} = (-i2\pi)^j \int_{-\infty}^\infty \xi^j \phi_h(\xi) e^{-i2\pi f\xi} d\xi; \quad (1.33)$$

hence, setting $f = 0$ in Eq. (1.33), we have

$$\int_{-\infty}^\infty \xi^j \phi_h(\xi) d\xi = \frac{1}{(-i2\pi)^j} \left. \frac{d^j |H(f)|^2}{df^j} \right|_{f=0}. \quad (1.34)$$

For $j = \text{odd}$, both sides of Eq. (1.34) vanish; however, for $j = \text{even}$, we have

$$\int_{-\infty}^\infty \xi^j \phi_h(\xi) d\xi = 2 \int_0^\infty \xi^j \phi_h(\xi) d\xi, \quad j = \text{even}. \quad (1.35)$$

Thus, the terms $j = \text{even}$ in Eq. (1.30) may be evaluated from $|H(f)|^2$ using Eqs. (1.34) and (1.35). Furthermore, for well-behaved $h(t)$, we have from the Fourier mate to Eq. (1.9):

$$\dot{h}(t) = \int_{-\infty}^{\infty} i2\pi f H(f) e^{i2\pi f t} df ; \quad (1.36)$$

hence, $i2\pi f H(f)$ is the Fourier transform of $\dot{h}(t)$. It follows from this fact and Eq. (1.34) that

$$\int_{-\infty}^{\infty} \xi^j \phi_{\dot{h}}(\xi) d\xi = \frac{4\pi^2}{(-i2\pi)^j} \frac{d^j}{df^j} |fH(f)|^2 \Big|_{f=0} . \quad (1.37)$$

For $j = \text{odd}$, both sides of Eq. (1.37) vanish; however, for $j = \text{even}$, we have

$$\int_{-\infty}^{\infty} \xi^j \phi_{\dot{h}}(\xi) d\xi = 2 \int_0^{\infty} \xi^j \phi_{\dot{h}}(\xi) d\xi , \quad j = \text{even}. \quad (1.38)$$

The terms $j = \text{even}$ in Eq. (1.31) may be evaluated from $|H(f)|^2$ using Eqs. (1.37) and (1.38).

For situations where odd powers of j are included in Eqs. (1.30) and (1.31), a different approach is available for evaluating these expressions. We may decompose $\phi_y(\tau)$ into components arising from $w_s(t)$ and $w_f(t)$ where, from Eq. (1.19a), we see that the contribution from $w_s(t)$ can be written as

$$\phi_{y_s}(\tau) = \int_{-\infty}^{\infty} \phi_h(\xi+\tau) \phi_{w_s}(\xi) d\xi . \quad (1.39)$$

Differentiating Eq. (1.39) twice yields

$$\phi_{y_s}''(\tau) = \int_{-\infty}^{\infty} \phi_h''(\xi+\tau) \phi_{w_s}(\xi) d\xi , \quad (1.40)$$

where we note that

$$E\{\dot{y}_S^2\} = -\phi_{y_S}''(0) = -\int_{-\infty}^{\infty} \phi_h''(\xi) \phi_{w_S}(\xi) d\xi, \quad (1.41)$$

see — e.g., p. 33 of Ref. 18. In regard to the behavior of $\phi_h''(\xi)$ at $\xi = 0$, we point out that inclusion of mass or inertia in the aircraft impulse response $h(t)$ guarantees continuity of $\phi_h''(\xi)$ at $\xi = 0$. Since $\phi_h(\xi)$ is an even function of ξ , $\phi_h''(\xi)$ also must be even; hence, we may rewrite Eq. (1.41) as

$$E\{\dot{y}_S^2\} = -2 \int_0^{\infty} \phi_h''(\xi) \phi_{w_S}(\xi) d\xi, \quad (1.42)$$

while, from Eq. (1.39), we may express $E\{y_S^2\} \equiv \phi_{y_S}(0)$ as

$$E\{y_S^2\} = 2 \int_0^{\infty} \phi_h(\xi) \phi_{w_S}(\xi) d\xi. \quad (1.43)$$

Combining Eqs. (1.43) and (1.42) with Eq. (1.29) yields

$$E\{y_S^2\} \approx 2 \sum_{j=0}^m a_j \int_0^{\infty} \xi^j \phi_h(\xi) d\xi, \quad (1.44)$$

and

$$E\{\dot{y}_S^2\} \approx -2 \sum_{j=0}^m a_j \int_0^{\infty} \xi^j \phi_h''(\xi) d\xi, \quad (1.45)$$

where Eqs. (1.44) and (1.45) are valid for cases where both odd and even powers are included in Eq. (1.29), or cases where only even powers are included. Furthermore, it is possible to avoid taking moments of the *derivatives* of $\phi_h(\xi)$ as we now show. Using integration by parts, we have

$$\begin{aligned} \int_0^{\infty} \xi^j \phi_h''(\xi) d\xi &= \xi^j \phi_h'(\xi) \Big|_0^{\infty} - j \int_0^{\infty} \xi^{j-1} \phi_h'(\xi) d\xi \\ &= -j \int_0^{\infty} \xi^{j-1} \phi_h'(\xi) d\xi, \quad j \geq 1, \end{aligned} \quad (1.46)$$

since the first term in the right-hand side of the first line vanishes for $j > 1$. Repeating the integration by parts on the right-hand side of Eq. (1.46), we further have

$$\begin{aligned} \int_0^{\infty} \xi^j \phi_h''(\xi) d\xi &= -j \left[\xi^{j-1} \phi_h(\xi) \right]_0^{\infty} - (j-1) \int_0^{\infty} \xi^{j-2} \phi_h(\xi) d\xi \\ &= j(j-1) \int_0^{\infty} \xi^{j-2} \phi_h(\xi) d\xi, \quad j \geq 2, \end{aligned} \quad (1.47)$$

since the first term in the right-hand side of the first line vanishes in this case for $j > 2$. Combining Eqs. (1.46) and (1.47) with Eq. (1.45) yields

$$\begin{aligned} E\{\dot{y}_s^2\} &\approx -2 \left[a_0 \int_0^{\infty} \phi_h''(\xi) d\xi - a_1 \int_0^{\infty} \phi_h'(\xi) d\xi \right. \\ &\quad \left. + \sum_{j=2}^m j(j-1) a_j \int_0^{\infty} \xi^{j-2} \phi_h(\xi) d\xi \right]. \end{aligned} \quad (1.48)$$

Equations (1.44) and (1.48) are particularly suitable for computing $E\{y_s^2\}$ and $E\{\dot{y}_s^2\}$ in cases where the "one-sided" expansion including odd powers of ξ is used in Eq. (1.29). In view of the fact that the first or higher order derivatives of turbulence autocorrelation functions may not be continuous at $\xi = 0$, representation of $\phi_{ws}(\xi)$ over the interval $0 \leq \xi \leq T_h$ by Eq. (1.29) using odd as well as even powers of ξ , and then computing $E\{y_s^2\}$ and $E\{\dot{y}_s^2\}$ with Eqs. (1.44) and (1.48) is probably the best overall method. It is easy to relate the moments in the right-hand sides of Eqs. (1.44) and (1.48) to derivatives of the unilateral Laplace transform of $\phi_h(\xi)$ — which may be useful in evaluating the moments.

The autocorrelation function $\phi_{ws}(\xi)$ of the "slow" component $w_s(t)$ contains complete information about the power spectrum of $w_s(t)$, because the two are a Fourier transform pair. The most useful general set of parametric descriptors of this Fourier transform pair appears to be the set of coefficients a_j , $j=0,1,\dots,m$ of a power series representation of $\phi_{ws}(\xi)$. Since $\phi_{ws}(\xi)$ is necessarily an even function of ξ ,

this representation may be chosen to approximate $\phi_{w_s}(\xi)$ over either $0 \leq \xi \leq T_h$ or $-T_h \leq \xi \leq T_h$, where T_h is the duration of the aircraft impulse response of interest. The representation over $0 \leq \xi \leq T_h$ generally will be preferable because of its more rapid convergence. In this case, Eqs. (1.44) and (1.48) may be used to compute $E\{y_s^2\}$ and $E\{\dot{y}_s^2\}$. However, the representation over the interval $-T_h \leq \xi \leq T_h$ permits evaluation of all integrals in Eqs. (1.30) and (1.31) directly from the squared aircraft frequency response magnitude $|H(f)|^2$ using Eqs. (1.34), (1.35), (1.37), and (1.38).

In addition to a representation of $\phi_{w_s}(\xi)$, we shall require the first few moments of the probability density of $w_s(t)$, say the first four, in order to verify the Gaussian property of $w_s(t)$. A method for computing these moments and forming an approximation to the first-order probability density of $w_s(t)$ is described in Sec. 6.4 of Ref. 19.

Characterization of stochastic intensity $\sigma_f(t)$ of the "fast" component $w_f(t)$. The locally stationary assumption of Eq. (1.8) of this report involves only turbulence characteristics. For this assumption to be satisfied, variations in the intensity $\sigma_f(t)$ of the fast component must be small over intervals comparable with the integral scale or nominal correlation interval of the component $z(t)$. Two additional locally stationary conditions were studied in Ref. 19. These additional two conditions are given by Eqs. (3.43) and (3.46) on p. 32 or Eqs. (5.9) and (5.10) on p. 54 of Ref. 19. Equations (5.9) and (5.10) are a statement of these conditions for engineering purposes, and are somewhat more easily satisfied than Eqs. (3.43) and (3.46). *The physical interpretation of the combination of these two additional conditions is that variations in $\sigma_f(t)$ must be negligible over durations comparable to the duration T_h of the aircraft impulse response function of interest.* Thus, satisfaction of these additional two locally stationary conditions depends on aircraft characteristics as well as the behavior of the turbulence component $\sigma_f(t)$.

For situations where these additional two locally stationary conditions are satisfied, it was shown in Secs. 4.1 to 4.4 of Ref. 19 that the first-order probability density function and threshold mean crossing rates of an aircraft response variable can be computed from the probability density function of σ_f^2 ; and furthermore, that the first few moments of the probability density of σ_f^2 provide the most important

parametric descriptions of $\sigma_f^2(t)$. Although a careful study of these additional locally stationary conditions involving recorded turbulence data and actual aircraft characteristics has not been made, we generally would expect these additional two locally stationary conditions to be satisfied for subsonic aircraft for engineering purposes. However, these additional two conditions may not be satisfied in the case of supersonic aircraft.

It will be shown in later sections of this report that when the additional two locally stationary conditions are not satisfied, we shall require the power spectral density $\Phi_{\sigma_f^2}(f)$ or autocorrelation function $\phi_{\sigma_f^2}(\tau)$ of $\sigma_f^2(t)$ to predict the most important nonGaussian correction terms for the first-order probability density of an aircraft response variable. The most important *parametric* descriptors of $\phi_{\sigma_f^2}(\tau)$ are the first few "one-sided" or "two-sided" power series expansion coefficients of $\phi_{\sigma_f^2}(\tau)$ — as was the case for $\phi_{w_s}(\tau)$.

Characterization of stationary Gaussian component $z(t)$ of the "fast" component $w_f(t)$. Since $z(t)$ is, by hypothesis, a stationary Gaussian process with zero mean value and unit variance, it is completely described by its power spectral density or autocorrelation function. In our computational work, we shall assume that $z(t)$ possesses the appropriate (transverse or longitudinal) von Karman spectral form. For these cases, $z(t)$ is completely described by a single parameter — the integral scale L_z of the appropriate transverse or longitudinal von Karman spectral form.

Summary of Turbulence Model Characterizations

Basic Model

$$w(t) = w_s(t) + \sigma_f(t) z(t)$$

$$\sigma_f(t) \geq 0, \quad E\{z\} = 0, \quad E\{z^2\} = 1,$$

$$w_s(t) \text{ stationary and Gaussian,}$$

$$\sigma_f(t) \text{ stationary,}$$

$z(t)$ stationary and Gaussian

$w_s(t)$, $\sigma_f(t)$, and $z(t)$ mutually independent.

"Complete" Characterization of Model

Component	Characterization
$w_s(t)$	P.S.D. or Autocorrelation function of $w_s(t)$, (Also P.D.F. of $w_s(t)$ to "check" Gaussian Assumption.)
$\sigma_f^2(t)$	P.S.D. or Autocorrelation Function of $\sigma_f^2(t)$, P.D.F. of $\sigma_f^2(t)$.
$z(t)$	P.S.D. or Autocorrelation Function of $z(t)$.

Parametric Characterization of Model

Component	Characterization
$w_s(t)$	"One-sided" or "two-sided" power series expansion coefficients of autocorrelation function of $w_s(t)$, (Also first few moments of P.D.F. of $w_s(t)$ to check Gaussian assumption.)
$\sigma_f^2(t)$	"One-sided" or "two-sided" power series expansion coefficients of autocorrelation function of $\sigma_f^2(t)$, First few moments of P.D.F. of $\sigma_f^2(t)$.
$z(t)$	Integral scale of appropriate (transverse or longitudinal) von Karman spectrum.

AIRCRAFT RESPONSE PROBABILITY DENSITY FUNCTIONS

In Sec. 4.4 of Ref. 19, a series expansion was developed for the first-order probability density function of an arbitrary aircraft response variable. The expansion is valid for situations where the three locally stationary conditions described above and in Ref. 19 are valid. In this section, a similar series expansion for the first-order probability density of an arbitrary aircraft response variable is developed; however, in this new treatment only the first locally stationary condition described by Eq. (1.8) is required. This condition depends on turbulence properties only, and is believed to be virtually always satisfied. Thus, the present results apply to supersonic aircraft with arbitrarily high Mach numbers — as well as to subsonic aircraft for which the simpler results of Sec. 4.4 of Ref. 19 apply.

Gaussian Property of Response Process Conditioned on the Intensity Modulation Process $\sigma_f(t)^*$

In Sec. 2, we shall assume that the "slow" component $w_s(t)$ and the component $z(t)$ in the turbulence model of Eq. (1.2) both are stationary Gaussian processes with zero mean values. *When $w_s(t)$ and $z(t)$ satisfy this zero mean and Gaussian assumption, the response process $y(t)$, conditioned on the process $\sigma_f(t)$ is a zero mean strictly Gaussian (generally nonstationary) process.* To prove this, we note first that each sample function of the response process $\{y(t)\}$ can be expressed as

$$y(t)|\sigma_f(u) = y_s(t) + y_f(t)|\sigma_f(u) , \quad -\infty < u \leq t \quad (2.1)$$

where $y_s(t)$ is the aircraft response to the "slow" turbulence sample function $w_s(t)$ in Eq. (1.2),

$$y_s(t) = \int_0^\infty h(\tau) w_s(t-\tau) d\tau , \quad (2.2)$$

*This section closely parallels Sec. 4.1 of Ref. 19.

and $y_f(t)|\sigma_f(u)$ is the aircraft response to the "fast" turbulence sample function $w_f(t)$ in Eq. (1.2),

$$y_f(t)|\sigma_f(u) = \int_0^{\infty} h(\tau) [w_f(t-\tau)|\sigma_f(u)] d\tau, \quad -\infty < u \leq t \quad (2.3a)$$

$$= \int_0^{\infty} h(\tau) \sigma_f(t-\tau) z(t-\tau) d\tau. \quad (2.3b)$$

The vertical bars followed by $\sigma_f(u)$ in Eqs. (2.1) and (2.3) denote that, in the stochastic process interpretation of these equations, the *sample function* $\sigma_f(u)$ is assumed to be known or specified for all values of u within the interval $-\infty < u \leq t$. Also, in writing Eqs. (2.2) and (2.3), we have assumed that $h(t)$ is deterministic and causal — i.e., that $h(t) = 0$ for $t < 0$. When $\sigma_f(u)$ is assumed to be known or specified for all $-\infty < u \leq t$, the right-hand side of Eq. (2.3b) represents a (deterministic) linear transformation of the Gaussian random function $z(t)$. Thus, the *conditional random process* $\{y_f(t)|\sigma_f(u)\}$, where $\sigma_f(u)$ is specified for all $-\infty < u \leq t$, is itself strictly Gaussian (and generally nonstationary) — since any linear transformation of a Gaussian process is itself Gaussian — e.g., Cramer [8], pp. 312 and 313. Furthermore, $\{w_s(t)\}$ is assumed to be Gaussian; thus, from the linearity of Eq. (2.2), the *random process* $\{y_s(t)\}$ also is Gaussian. Moreover, since $\{w_s(t)\}$ and $\{z(t)\}$ are assumed to be independent, $\{y_s(t)\}$ and $\{y_f(t)|\sigma_f(u)\}$ also are independent. Finally, since the sum of any number of independent Gaussian processes is necessarily Gaussian — e.g., Cramer [8], p. 316 — it follows from Eq. (2.1) that the conditional response process $\{y(t)|\sigma_f(u)\}$, where $\sigma_f(u)$ is specified for all $-\infty < u \leq t$, is strictly Gaussian (and generally nonstationary). This result does not depend on any locally stationary assumption. From the zero mean value assumptions for the processes $\{w_s(t)\}$ and $\{z(t)\}$ it follows further from Eqs. (2.2) and (2.3b) that $\{y(t)|\sigma_f(u)\}$ also has zero mean value.

Let us denote the *conditional* mean square value of the process $\{y(t)\}$ by

$$\sigma_y^2 = \sigma_y^2(t) \triangleq E\{y^2(t)|\sigma_f(u)\}, \quad -\infty < u \leq t; \quad (2.4)$$

that is, σ_y^2 is the expected value of the squared system response given that $\sigma_f(u)$ is specified for all $-\infty < u \leq t$. The expectation operation therefore takes place over the ensembles of input functions $\{w_s(t)\}$ and $\{z(t)\}$ with $\sigma_f(u)$ being considered as known. Since $\{y(t)|\sigma_f(u)\}$ is generally non-stationary, σ_y^2 is generally a function of t . Furthermore, when we (later) consider $\sigma_f(t)$ to be a stochastic function of time, $\sigma_y^2(t)$ also becomes a stochastic function of time.

Let us now consider the *function* $\sigma_f(u)$, $-\infty < u \leq t$ to be the limiting case of an infinite dimensional "vector" $\underline{g}_f \equiv \sigma_f(u_1, u_2, \dots, u_n)$ as n is taken to approach infinity and $u_{j+1} - u_j$ is shrunk to zero for all $j = 1, 2, \dots, n-1$. Thus, the assumption that the "vector" \underline{g}_f is specified is identical to an assumption that the function $\sigma_f(u)$ is specified for all $-\infty < u \leq t$. Let $p(y|\underline{g}_f)$ denote the conditional probability density of the aircraft response $y(t)$ given that $\sigma_f(u)$ is specified for all $-\infty < u \leq t$. Then, from the above discussion, $p(y|\underline{g}_f)$ is strictly normally distributed with variance $\sigma_y^2 = \sigma_y^2(t)$ described by Eq. (2.4):

$$p(y|\underline{g}_f) = \frac{1}{\sqrt{2\pi} \sigma_y} e^{-\frac{y^2}{2\sigma_y^2}} \quad (2.5)$$

Series Expansion of Response Probability Density Functions

Equation (2.5) expresses the conditional probability density of the aircraft response $y(t)$ given that the random function $\sigma_f(u)$ is specified for all values of $0 < u \leq t$. The unconditional probability density of the aircraft response is the expectation of $p(y|\underline{g}_f)$ with respect to the joint probability density of the vector \underline{g}_f ; i.e.,

$$p(y) = \int_{\underline{g}}^{\infty} p(y|\underline{g}_f) p(\underline{g}_f) d\underline{g}_f \quad (2.6a)$$

$$= E\{p(y|\underline{g}_f)\} \quad , \quad (2.6b)$$

where the integral in Eq. (2.6) represents the limiting case of an infinite order integral over the n -dimensional space $\{u_1, u_2, \dots, u_n\}$ as n is taken to approach infinity and $u_{j+1} - u_j$ is shrunk to zero for all $j = 1, 2, \dots, n-1$ as is described above. (It will become evident shortly that no such integrals will have to be carried out to apply the methods being presented.)

In most turbulence records, *fluctuations* in the stochastic function $\sigma_f(t)$ are of the order of not more than 50% of the mean value of $\sigma_f(t)$. Each sample function $\sigma_f(t)$ in our conceptual ensemble $\{\sigma_f(t)\}$ gives rise to a different function $\sigma_y^2(t)$ as indicated by Eq. (2.4). Furthermore, fluctuations in $\sigma_y^2(t)$ relative to the mean value of σ_y^2 are comparable to or less than fluctuations in $\sigma_f^2(t)$ relative to its mean. Consequently, we need only consider variations in the right-hand side of Eq. (2.5) that are caused by "small" variations in σ_y^2 relative to the mean $\overline{\sigma_y^2}$ of σ_y^2 :

$$\overline{\sigma_y^2} \triangleq E\{\sigma_y^2\} . \quad (2.7)$$

Such variations in $p(y|\sigma_f)$ may be efficiently represented by a Taylor's series expansion of the right-hand side of Eq. (2.5) in the variable σ_y^2 about its mean value $\overline{\sigma_y^2}$ - i.e.,

$$p(y|\sigma_f) = \sum_{k=0}^{\infty} \frac{p^{(k)}(y|\sigma_f)}{k!} (\sigma_y^2 - \overline{\sigma_y^2})^k , \quad (2.8)$$

where we have used the definition

$$p^{(k)}(y|\sigma_f) \triangleq \frac{d^k}{d(\sigma_y^2)^k} p(y|\sigma_f) \Big|_{\sigma_y^2 = \overline{\sigma_y^2}} \quad (2.9a)$$

$$= \frac{d^k}{d(\sigma_y^2)^k} \frac{1}{\sqrt{2\pi} \sigma_y^2} e^{-\frac{y^2}{2\sigma_y^2}} \Big|_{\sigma_y^2 = \overline{\sigma_y^2}} , \quad (2.9b)$$

according to Eq. (2.5), and where

$$p^{(0)}(y|\underline{\sigma}_f) = p(y|\underline{\sigma}_f) \Big|_{\sigma_y^2 = \overline{\sigma_y^2}}. \quad (2.10)$$

Let us denote the central moments of σ_y^2 by

$$\mu_{\sigma_y^2}^{(k)} \triangleq E\{(\sigma_y^2 - \overline{\sigma_y^2})^k\} \quad (2.11a)$$

$$= \int_0^\infty [E\{y^2(t)|\underline{\sigma}_f\} - \overline{\sigma_y^2}]^k p(\underline{\sigma}_f) d\underline{\sigma}_f, \quad (2.11b)$$

where, in going to the second line, we have used Eq. (2.4) and the vector notation for $\sigma_f(u)$. If we now substitute Eq. (2.8) into Eq. (2.6a), then interchange orders of integration and summation, and compare the resulting expression with Eq. (2.11), we see that our series expansion of $p(y)$ may be expressed as

$$p(y) = \sum_{k=0}^{\infty} \frac{\mu_{\sigma_y^2}^{(k)}}{k!} p^{(k)}(y|\underline{\sigma}_f), \quad (2.12)$$

where $p^{(k)}(y|\underline{\sigma}_f)$ is defined by Eq. (2.9) and $\mu_{\sigma_y^2}^{(k)}$ is the k th central moment of σ_y^2 as defined by Eq. (2.11).

Equation (2.12) is the desired series expansion of $p(y)$. Since $\mu_{\sigma_y^2}^{(0)} \equiv 1$, the first term in the right-hand side of Eq. (2.12) is $p(y|\underline{\sigma}_f) \Big|_{\sigma_y^2 = \overline{\sigma_y^2}}$, whereas the term corresponding to $k = 1$ is zero because $\mu_{\sigma_y^2}^{(1)} \equiv 0$. Consequently, the first two nonvanishing terms of the right-hand side of Eq. (2.12) are

$$p(y) \sim p(y|\sigma_f) \Big|_{\sigma_y^2 = \overline{\sigma_y^2}} + \frac{1}{2} \mu_{\sigma_y^2}^{(2)} p^{(2)}(y|\sigma_f) . \quad (2.13)$$

Our motivation for taking $\sigma_y^2 = \overline{\sigma_y^2}$ as the expansion point of our Taylor's series is threefold: in this case (i) when σ_y^2 is a constant, the first term is in exact agreement with the known Gaussian result, (ii) the term $k = 1$ in Eq. (2.12) vanishes identically since $\mu_{\sigma_y^2}^{(1)} \equiv 0$, and (iii) the second moment of σ_y^2 is minimized about the expansion point $\sigma_y^2 = \overline{\sigma_y^2}$; hence, the first correction term to the term $k = 0$ is minimized.

Discussion. From Eq. (2.5), we see that the first term in the right-hand side of Eq. (2.13) is the Gaussian density function with variance $\sigma_y^2 = \overline{\sigma_y^2}$. Thus, in those situations where σ_y^2 is a constant — which occur when σ_f^2 is a constant — Eq. (2.13) reduces to the known Gaussian result for stationary Gaussian aircraft excitations. In Appendix A, it is shown that the low-order correction terms to the Gaussian first term in Eq. (2.12) are

$$p^{(1)}(y|\sigma_f) = \frac{p(y|\sigma_f)}{2\sigma_y^2} \left[\frac{y^2}{\sigma_y^2} - 1 \right] \Big|_{\sigma_y^2 = \overline{\sigma_y^2}} \quad (2.14a)$$

$$= \frac{p(y|\sigma_f)}{2\sigma_y^2} H_2\left(\frac{y}{\sigma_y}\right) \Big|_{\sigma_y = \sqrt{\overline{\sigma_y^2}}} , \quad (2.14b)$$

$$p^{(2)}(y|\sigma_f) = \frac{p(y|\sigma_f)}{4(\sigma_y^2)^2} \left[\frac{y^4}{(\sigma_y^2)^2} - 6 \frac{y^2}{\sigma_y^2} + 3 \right] \Big|_{\sigma_y^2 = \overline{\sigma_y^2}} \quad (2.15a)$$

$$= \frac{p(y|\sigma_f)}{4(\sigma_y^2)^2} H_4 \left(\frac{y}{\sigma_y} \right) \Big|_{\sigma_y = \sqrt{\overline{\sigma_y^2}}}, \quad (2.15b)$$

and

$$p^{(3)}(y|\sigma_f) = \frac{p(y|\sigma_f)}{8(\sigma_y^2)^3} \times \left[\frac{y^6}{(\sigma_y^2)^3} - 15 \frac{y^4}{(\sigma_y^2)^2} + 45 \frac{y^2}{\sigma_y^2} - 15 \right] \Big|_{\sigma_y^2 = \overline{\sigma_y^2}} \quad (2.16a)$$

$$= \frac{p(y|\sigma_f)}{8(\sigma_y^2)^3} H_6 \left(\frac{y}{\sigma_y} \right) \Big|_{\sigma_y = \sqrt{\overline{\sigma_y^2}}}. \quad (2.16b)$$

The general form of the above three terms is

$$p^{(k)}(y|\sigma_f) = \frac{p(y|\sigma_f)}{(2\sigma_y^2)^k} H_{2k} \left(\frac{y}{\sigma_y} \right) \Big|_{\sigma_y = \sqrt{\overline{\sigma_y^2}}}, \quad (2.17)$$

where, in Eqs. (2.14) through (2.17), $H_{2k}(\cdot)$ denotes the Hermite polynomial of degree $2k$ as defined on p. 133 of Ref. 8.

It may be shown that (at least through the term $k = 3$) the series expansion of Eq. (2.12) is identical to the Gram-Charlier series of type A - e.g., Ref. 8, pp. 222, 223. However, our derivation of Eq. (2.12) is entirely different from the usual derivation of the Gram-Charlier series. Our

derivation was based on a *Taylor's series* expansion of the Gaussian density of Eq. (2.5) where derivatives were taken with respect to the variance of σ_y^2 at a generic point y . Consequently, the expansion functions of Eq. (2.9) have the property that for sufficiently small low-order central moments $\mu_{\sigma_y^2}^{(k)}$, $k = 2, 3, \dots$, a small number of terms in the series of Eq. (2.12) should provide an excellent approximation to $p(y)$ for every value of y in the range $-\infty < y < \infty$. That is, the forms of the expansion functions $p^{(k)}(y|\sigma_f)$, $k = 2, 3, \dots$, (considered as functions of y) are the optimal choices for a good fit over the entire range $-\infty < y < \infty$ of interest of the variable y . A property of this type cannot be inferred from common derivations of the Gram-Charlier — e.g., Ref. 8, pp. 222, 223 — and, in fact, may be valid only for linear transformations of the class of intensity modulated Gaussian processes being considered herein — cf., von Mises [32, p. 137].

Example of Two-Term Expansion of Response Probability Density Functions

In order to evaluate the terms through, say, $k = N$ in the expansion of Eq. (2.8), we require $\overline{\sigma_y^2} = E\{\sigma_y^2\}$ and the central moments defined by Eq. (2.11a) through $k = N$. The quantity $E\{\sigma_y^2\}$ can be evaluated by integrating over $-\infty < f < \infty$ the power spectral density of the aircraft response. In the next section of this report, a new method will be described for evaluating the coefficient $\mu_{\sigma_y^2}^{(2)}$ of the first and most important correction term in the expansion. The accuracy of the two terms of the expansion described by Eq. (2.13) therefore is of considerable interest.

To ascertain typical accuracy that can be expected from the two-term expansion described by Eq. (2.13), we consider a one-dimensional analog of Eq. (2.6a), which, we recall is the exact expression for $p(y)$ that Eq. (2.13) approximates. To relate this one-dimensional analog to the infinite dimensional integration in Eq. (2.6a), we recall that when fluctuations in the process $\sigma_f(t)$ are negligible over time intervals comparable with the nominal duration of the aircraft impulse response $h(t)$, the response process is locally stationary and Gaussian [19] with instantaneous time-varying variance

$$\sigma_y^2 \equiv \sigma_y^2(t) \approx E\{y^2(t)|\sigma_f(t)\} \quad . \quad (2.18)$$

Equation (2.18) differs from Eq. (2.4) in that in the quasi-stationary approximation involved in Eq. (2.18), at each value of t , $\sigma_y^2(t)$ depends on the value of the function $\sigma_f(t)$ observed only at the same instant of time — rather than on $\sigma_f(u)$ observed over $-\infty < u \leq t$ as in Eq. (2.4). Thus, Eq. (2.6a) may be replaced in this quasi-stationary approximation by

$$p(y) = \int_0^\infty p(y|\sigma_f) p(\sigma_f) d\sigma_f \quad (2.19a)$$

$$= \int_0^\infty \frac{1}{\sqrt{2\pi\sigma_y^2}} e^{-\frac{y^2}{2\sigma_y^2}} p(\sigma_f) d\sigma_f \quad (2.19b)$$

$$= \int_0^\infty \frac{1}{\sqrt{2\pi\sigma_y^2}} e^{-\frac{y^2}{2\sigma_y^2}} p(\sigma_y^2) d\sigma_y^2, \quad (2.19c)$$

where,* in going to Eq. (2.19b) we have substituted Eq. (2.5) where σ_y^2 is a function of σ_f as indicated by Eq. (2.18), and in going to Eq. (2.19c) the probability density $p(\sigma_f)$ has been mapped into the probability density $p(\sigma_y^2)$ with the transformation between σ_f and σ_y^2 described by Eq. (2.18). In this regard, we note that Eq. (2.18) implies the existence of a generally different value of σ_y^2 for each different value of σ_f — i.e., defines σ_y^2 as a function of σ_f . Therefore, the probability density $p(\sigma_f)$ implicitly defines from the function $\sigma_y^2(\sigma_f)$ the probability density $p(\sigma_y^2)$ shown in Eq. (2.19c).

When the aircraft excitation is a stationary Gaussian process, σ_f is a constant; therefore, from Eq. (2.18) σ_y^2 also is a constant and $p(\sigma_y^2)$ is a delta function located at the correct value of σ_y^2 . For this limiting case, the integration in Eq. (2.19c) will yield a Gaussian probability density with the correct variance.

*In Eqs. (2.19a) to (2.19c) and the discussion following them, we denote the probability density functions of the random variables σ_f and σ_y^2 by $p(\sigma_f)$ and $p(\sigma_y^2)$, respectively. Therefore, $p(\sigma_f)$ and $p(\sigma_y^2)$ are different functions of their arguments. To keep the notational problem from getting out of control, we shall generally follow this practice in the following pages. That is, probability density functions of different random variables will be denoted by $p(\cdot)$ with the arguments being the random variables the densities describe.

In order to ascertain the typical accuracy that can be expected from the two-term expansion of Eq. (2.13), we consider for the probability density $p(\sigma_y^2)$ in Eq. (2.19c) the gamma density - e.g., pp. 220-221 of Ref. 33:

$$p(\sigma_y^2) = \begin{cases} \frac{\gamma}{\sigma_y^2 \Gamma(\gamma)} \left(\gamma \sigma_y^2 / \overline{\sigma_y^2} \right)^{\gamma-1} e^{-\gamma \sigma_y^2 / \overline{\sigma_y^2}}, & \sigma_y^2 \geq 0 \\ 0 & , \quad \sigma_y^2 < 0 . \end{cases} \quad (2.20)$$

The gamma density function generally is expressed as a function of two "free parameters" which are uniquely determined by the mean and variance of the distribution. Instead, we have written Eq. (2.20) directly in terms of the mean $\overline{\sigma_y^2}$ of the distribution and one additional free parameter. This remaining free parameter is the reciprocal of the relative variance of σ_y^2 - i.e.,

$$\gamma \equiv \frac{(\overline{\sigma_y^2})^2}{E\{(\sigma_y^2 - \overline{\sigma_y^2})^2\}} \quad (2.21a)$$

$$= \frac{1}{\text{Relative variance of } \sigma_y^2} . \quad (2.21b)$$

Thus, when $\gamma \rightarrow \infty$, the density described by Eq. (2.20) approaches a delta function located at $\sigma_y^2 = \overline{\sigma_y^2}$. On the other hand, when $\gamma = 1$, Eq. (2.20) describes the exponential probability density. For large finite values of γ , it can be shown that Eq. (2.20) approaches a Gaussian density in the neighborhood of $\sigma_y^2 = \overline{\sigma_y^2}$.

In Fig. 7, the gamma probability density is shown plotted in the usual manner for several values of the parameter γ — see, for example, pp. 248 and 404 of Ref. 32. However, we have used our notation of Eq. (2.20) in describing the abscissa and ordinate in Fig. 7. When plotted as a function of the normalized variable of $\sigma_y^2/\bar{\sigma}_y^2$ that is "natural" to the gamma density as described by Eq. (2.20), we obtain the behavior shown in Fig. 8. Notice that each density function shown in Fig. 8 has the same mean value, and that Fig. 8 shows the relative variance $1/\gamma$ shrinking as γ increases — as we have described above. In the limit $\gamma \rightarrow \infty$ $p(\sigma_y^2)$ approaches a Dirac delta function located at $\sigma_y^2 = \bar{\sigma}_y^2$. Figure 8 displays Eq. (2.20) in the form relevant to the present work — the plots shown in Fig. 7 are included only for comparison with the gamma density as it is usually shown. From Fig. 8, we can see that the gamma density function of Eq. (2.20) encompasses a nice range of shapes to model the probability density of σ_y^2 for purposes of studying the accuracy of the two-term expansion of $p(y)$ given by Eq. (2.13). Fortunately, the exact density function $p(y)$ also can be evaluated in closed form when the gamma density of Eq. (2.20) is substituted into Eq. (2.19c) and the integration carried out.

When Eqs. (2.19c) and (2.20) are combined, it is shown in Appendix B that we may express the resulting probability density in terms of the normalized response variable

$$n \triangleq \frac{y}{\sqrt{\sigma_y^2}} \quad (2.22)$$

as

$$\begin{aligned} p_\gamma(n) &\triangleq p(y/\sqrt{\sigma_y^2}) \\ &= \frac{\sqrt{2\gamma/\pi}}{2^{\gamma-1/2}\Gamma(\gamma)} (\sqrt{2\gamma}|n|)^{\gamma-1/2} K_{\gamma-1/2}(\sqrt{2\gamma}|n|) \quad , \end{aligned} \quad (2.23)$$

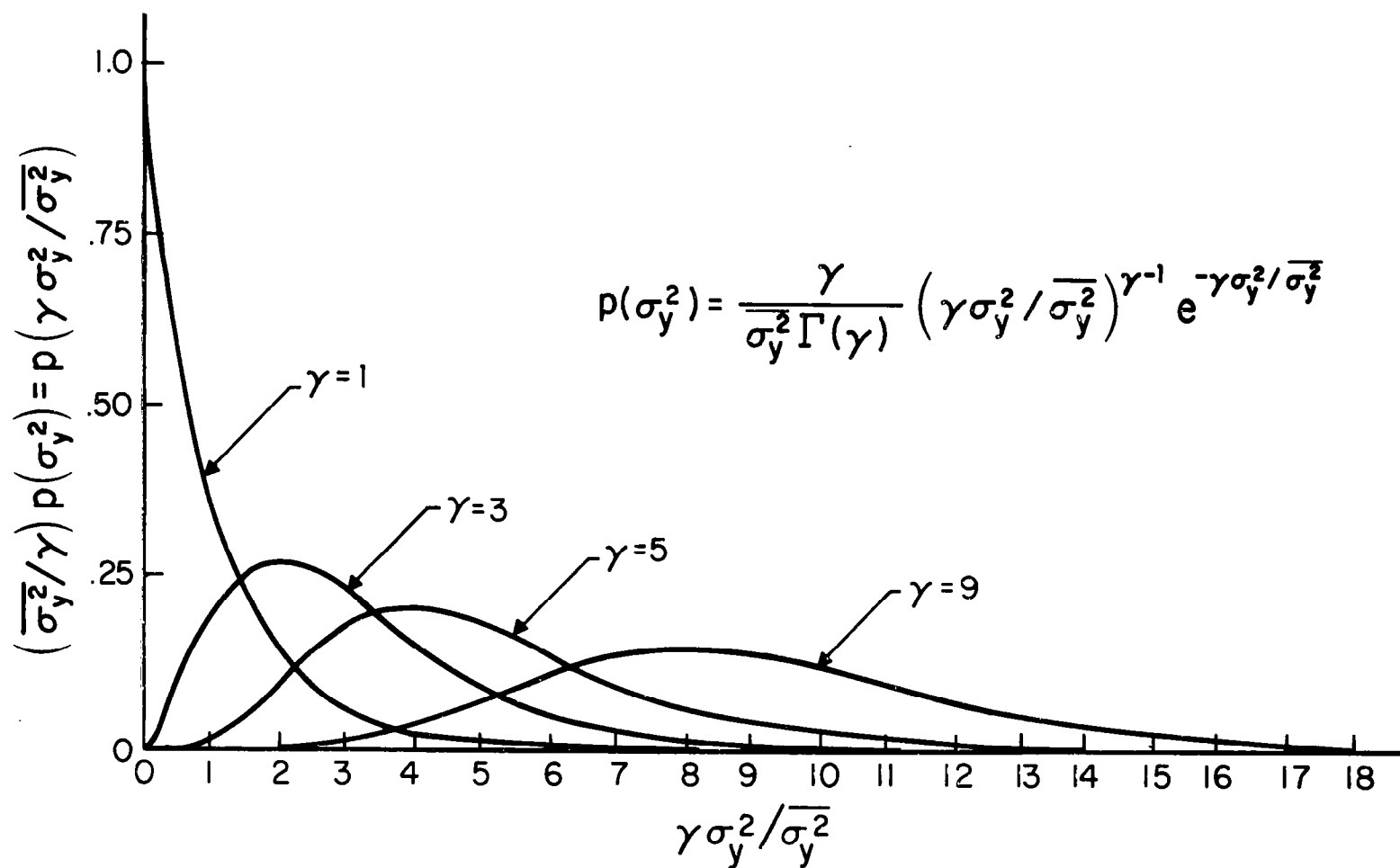


FIG. 7. GAMMA PROBABILITY DENSITY OF EQ. (2.20) PLOTTED AS A FUNCTION OF $\gamma \sigma_y^2 / \sigma_y^2$

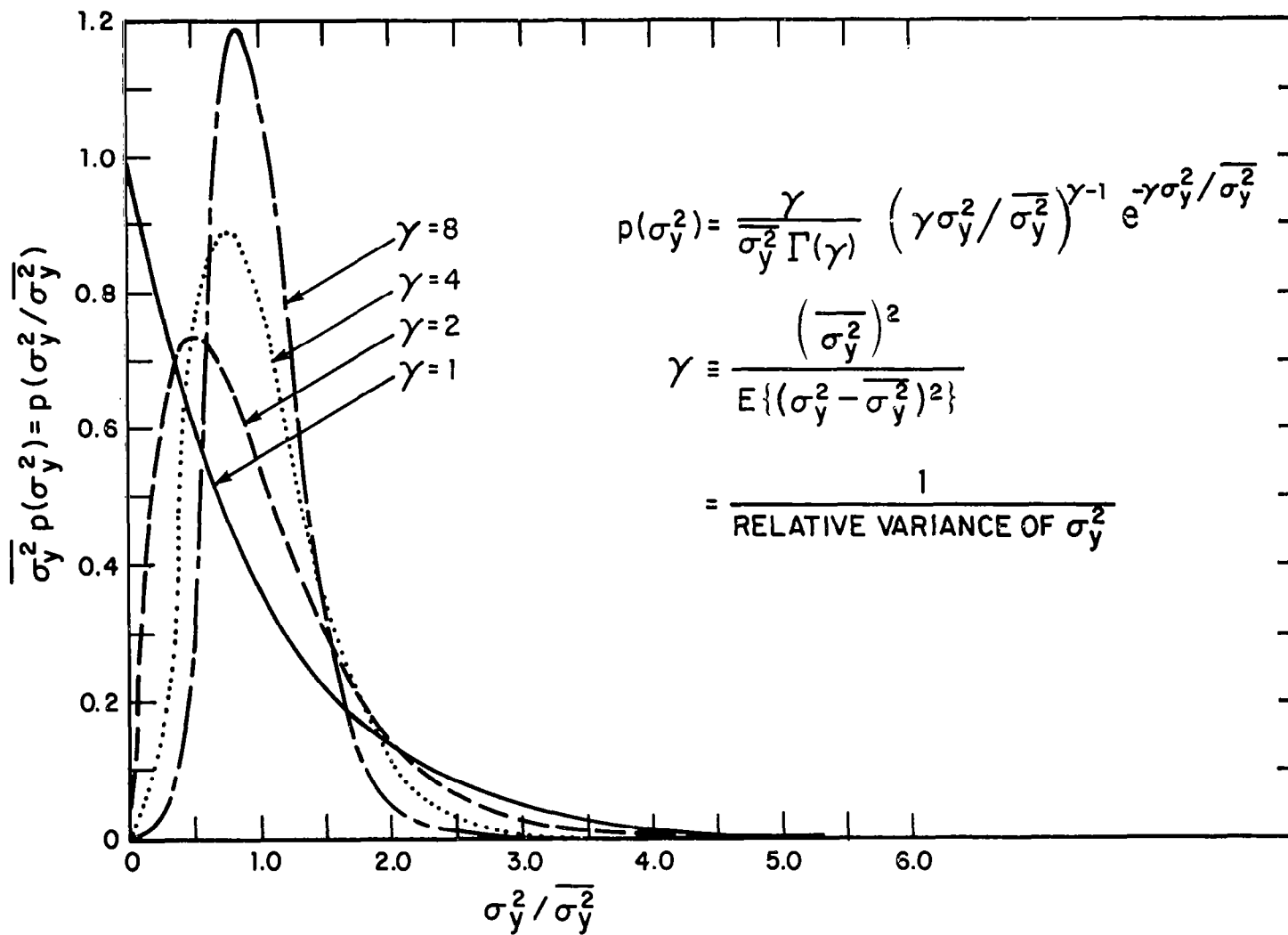


FIG. 8. GAMMA PROBABILITY DENSITY OF EQ. (2.20) PLOTTED AS A FUNCTION OF $\sigma_y^2 / \bar{\sigma}_y^2$.

where $K_n(\cdot)$ is the modified Bessel function of the second kind or order n . For $n = 1/2, 3/2, 5/2, \dots$, these Bessel functions can be expressed in closed form in terms of elementary functions. Therefore, whenever γ is a positive integer, it follows from Eq. (2.23) that $p_\gamma(\eta)$ can be expressed in closed form in terms of elementary functions. $K_n(\cdot)$ for $n = 1/2, 3/2, 5/2, \dots, 19/2$ are developed in Appendix C along with expressions for $p_\gamma(\eta)$ for $\gamma = 1, 2, 4, 8$, and 9.

Plots of the probability density $p(y/\sqrt{\sigma_y^2})$ given by Eq. (2.23) and evaluated from the results in Appendix C are shown in Figs. 9 to 12 for values of γ of 1, 2, 4, and 8 respectively. To compare these results with those produced by the two-term expansion of Eq. (2.13), we note first from Eqs. (2.5), (2.13), and (2.15), that our two-term approximation to $p(y)$ can be expressed as

$$p(y) \approx \frac{1}{\sqrt{2\pi\sigma_y^2}} e^{-\frac{y^2}{2\sigma_y^2}} \left\{ 1 + \frac{\mu_{\sigma_y^2}^{(2)}}{8(\sigma_y^2)^2} \left[\frac{y^4}{(\sigma_y^2)^2} - 6 \frac{y^2}{\sigma_y^2} + 3 \right] \right\}, \quad (2.24)$$

where from Eqs. (2.11a) and (2.21a), we see that the coefficient to the correction term may be expressed in terms of γ :

$$\frac{\mu_{\sigma_y^2}^{(2)}}{(\sigma_y^2)^2} \equiv \frac{1}{\gamma}. \quad (2.25)$$

Hence, when we introduce the normalized response variable of Eq. (2.22) our two-term series approximation of Eqs. (2.13) and (2.24) becomes

$$p(\eta) = p(y/\sqrt{\sigma_y^2}) \\ \approx \frac{1}{\sqrt{2\pi}} e^{-\frac{\eta^2}{2}} \left[1 + \frac{1}{8\gamma} (\eta^4 - 6\eta^2 + 3) \right]. \quad (2.26)$$

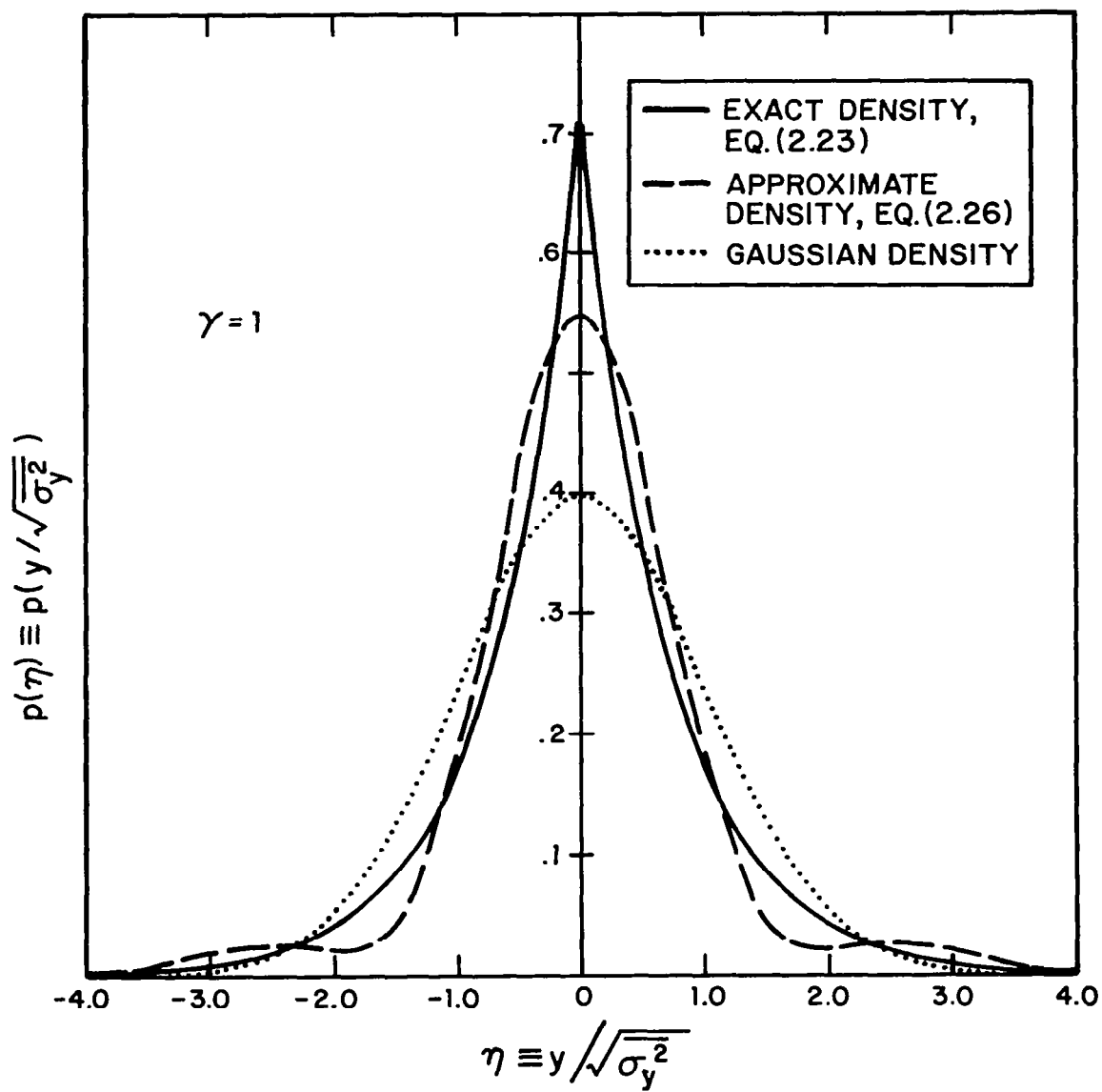


FIG. 9. COMPARISON OF EXACT AND APPROXIMATE PROBABILITY DENSITY FUNCTIONS, EQS. (2.23) AND (2.26) RESPECTIVELY, FOR $\gamma = 1$. DEFINITION OF γ IS GIVEN IN FIG. 8.

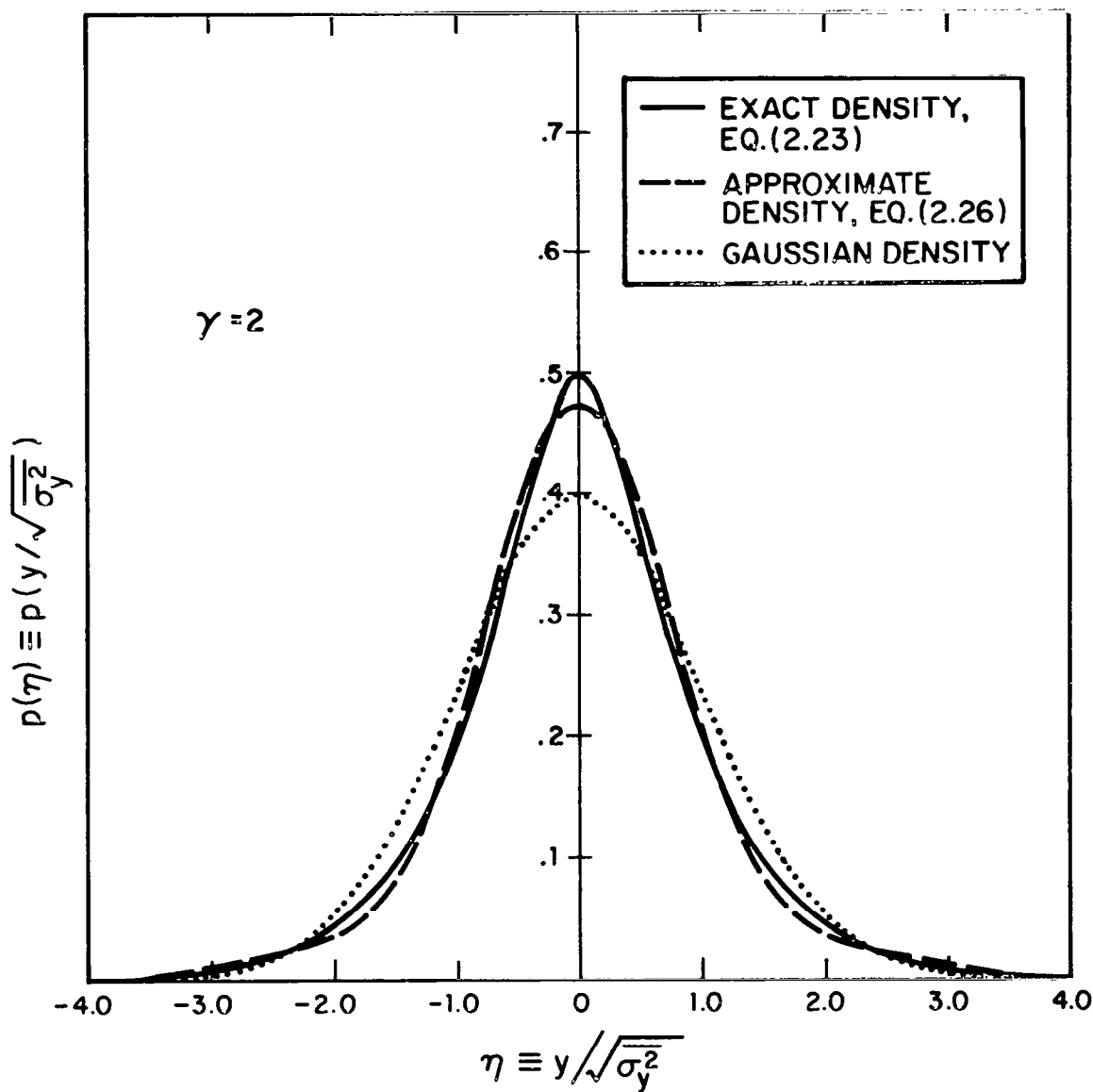


FIG. 10. COMPARISON OF EXACT AND APPROXIMATE PROBABILITY DENSITY FUNCTIONS, EQS. (2.23) AND (2.26) RESPECTIVELY, FOR $\gamma = 2$. DEFINITION OF γ IS GIVEN IN FIG. 8.

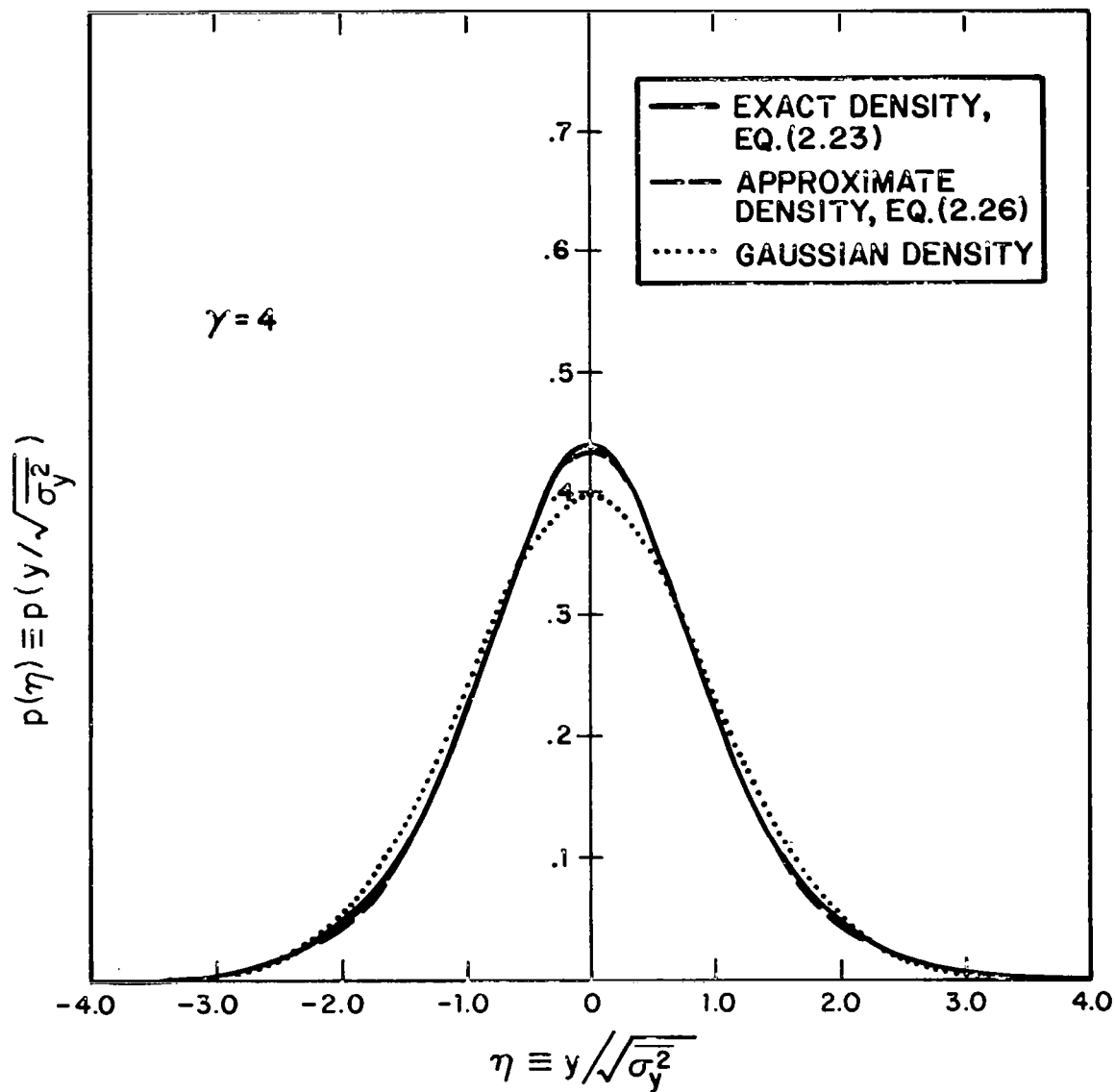


FIG. 11. COMPARISON OF EXACT AND APPROXIMATE PROBABILITY DENSITY FUNCTIONS, EQS. (2.23) AND (2.26) RESPECTIVELY, FOR $\gamma = 4$. DEFINITION OF γ IS GIVEN IN FIG. 8.

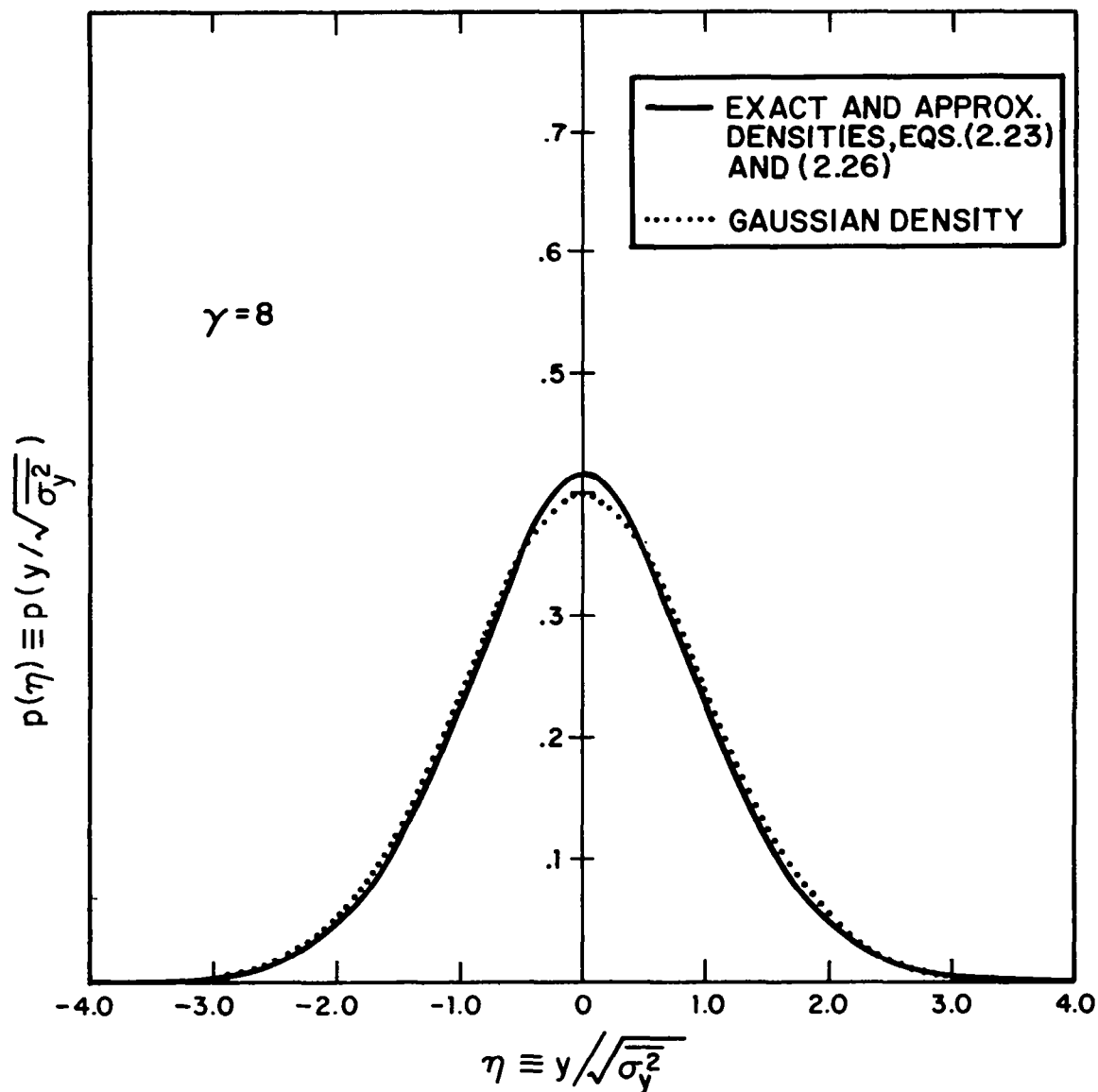


FIG. 12. COMPARISON OF EXACT AND APPROXIMATE PROBABILITY DENSITY FUNCTIONS, EQS. (2.23) AND (2.26) RESPECTIVELY, FOR $\gamma = 8$. DEFINITION OF γ IS GIVEN IN FIG. 8.

Plots of Eq. (2.26) are shown in Figs. 9 to 12 for comparison with the exact result given by Eq. (2.23). Also shown in each figure is the Gaussian approximation to the density function — i.e., the first term in Eq. (2.26). Examination of Figs. 9 to 12 shows that the two-term approximation of Eq. (2.26) provides marginal results for $\gamma = 1$, good results for $\gamma = 2$, and for $\gamma = 4$ and larger, the approximation is excellent.

We can express gamma in terms of the *relative standard deviation* of σ_y^2 — also called the *coefficient of variation* [8, p. 357] — i.e., from Eq. (2.21) we have

$$\frac{1}{\sqrt{\gamma}} = \frac{\sqrt{E\{(\sigma_y^2 - \overline{\sigma_y^2})^2\}}}{\overline{\sigma_y^2}} \quad (2.27a)$$

$$\equiv \text{coefficient of variation of } \sigma_y^2. \quad (2.27b)$$

Consequently, we see from Figs. 9 to 12 that when the coefficient of variation of the instantaneous mean-square response $\sigma_y^2(t)$ is unity, our two-term series approximation Eq. (2.24) provides only marginal accuracy; when the coefficient variation is $(1/\sqrt{2}) = 0.707$ the accuracy is good, and when the coefficient variation of the instantaneous mean-square response is $(1/\sqrt{4}) = 0.5$ or less, the error involved in the two-term approximation Eq. (2.24) is, for practical purposes, negligible.

We remind the reader that the probability density function of $\sigma_y^2/\overline{\sigma_y^2}$ used to generate the results of Figs. 9 to 12 are displayed in Fig. 8. Thus, for all but extremely strong variations in $\sigma_y^2(t)$, the two-term expansion, Eq. (2.24) will provide excellent results. To illustrate the excellent fit of Eq. (2.24) in the tails of the probability density of $p(y/\sqrt{\overline{\sigma_y^2}})$ the same curves shown in Figs. 9 to 12 are plotted in Figs. 13 to 16 on semi-logarithmic coordinates.

Relationship for Expansion Coefficient of Correction Term to the Gaussian Density

Let us turn now to obtaining a general relationship for $\mu_{\sigma_y^2}^{(2)}$ for cases where no locally stationary assumptions are required other than that described by Eq. (1.8) — which

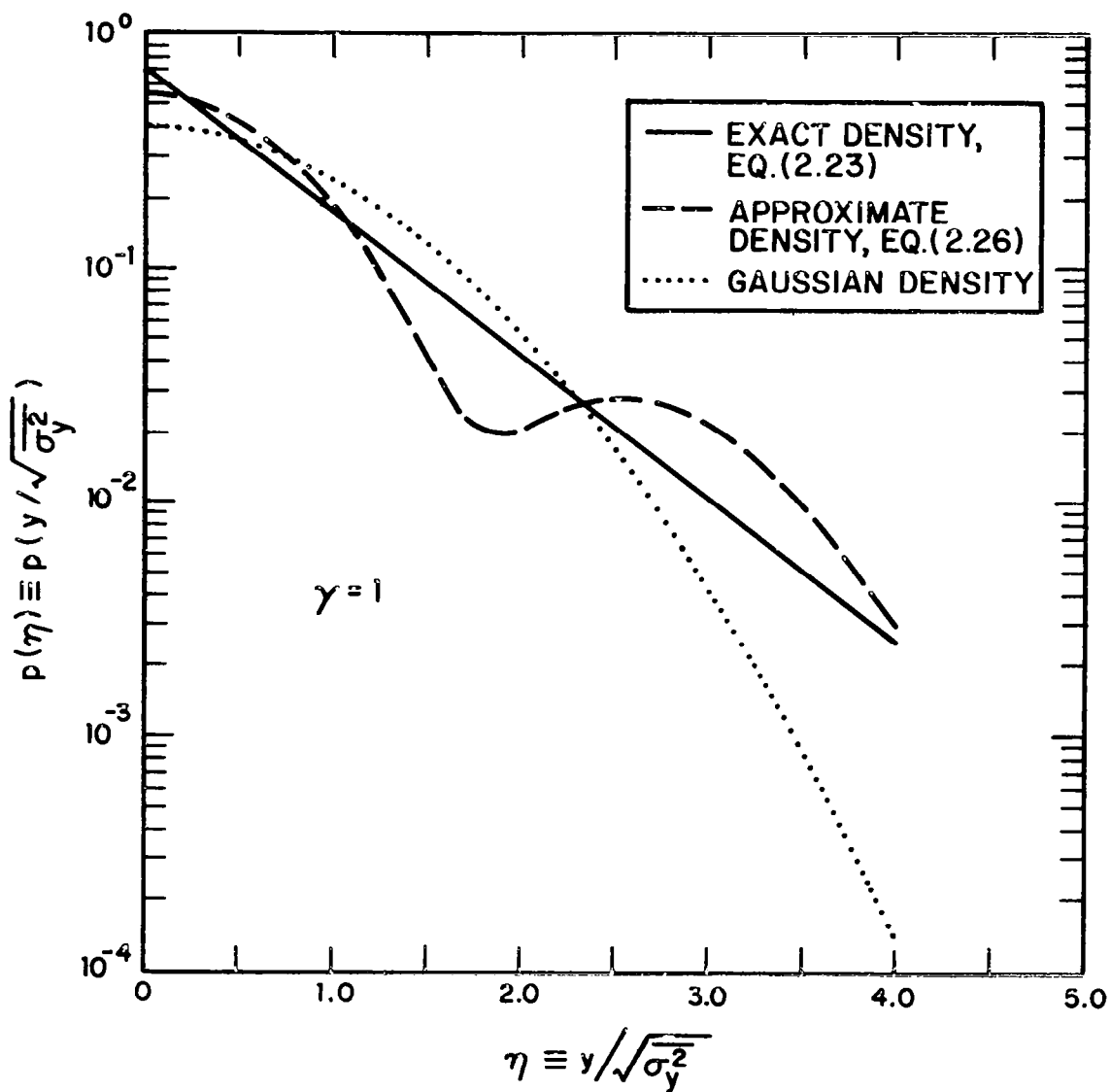


FIG. 13. COMPARISON OF EXACT AND APPROXIMATE PROBABILITY DENSITY FUNCTIONS, EQS. (2.23) AND (2.26) RESPECTIVELY, FOR $\gamma = 1$. DEFINITION OF γ IS GIVEN IN FIG. 8.

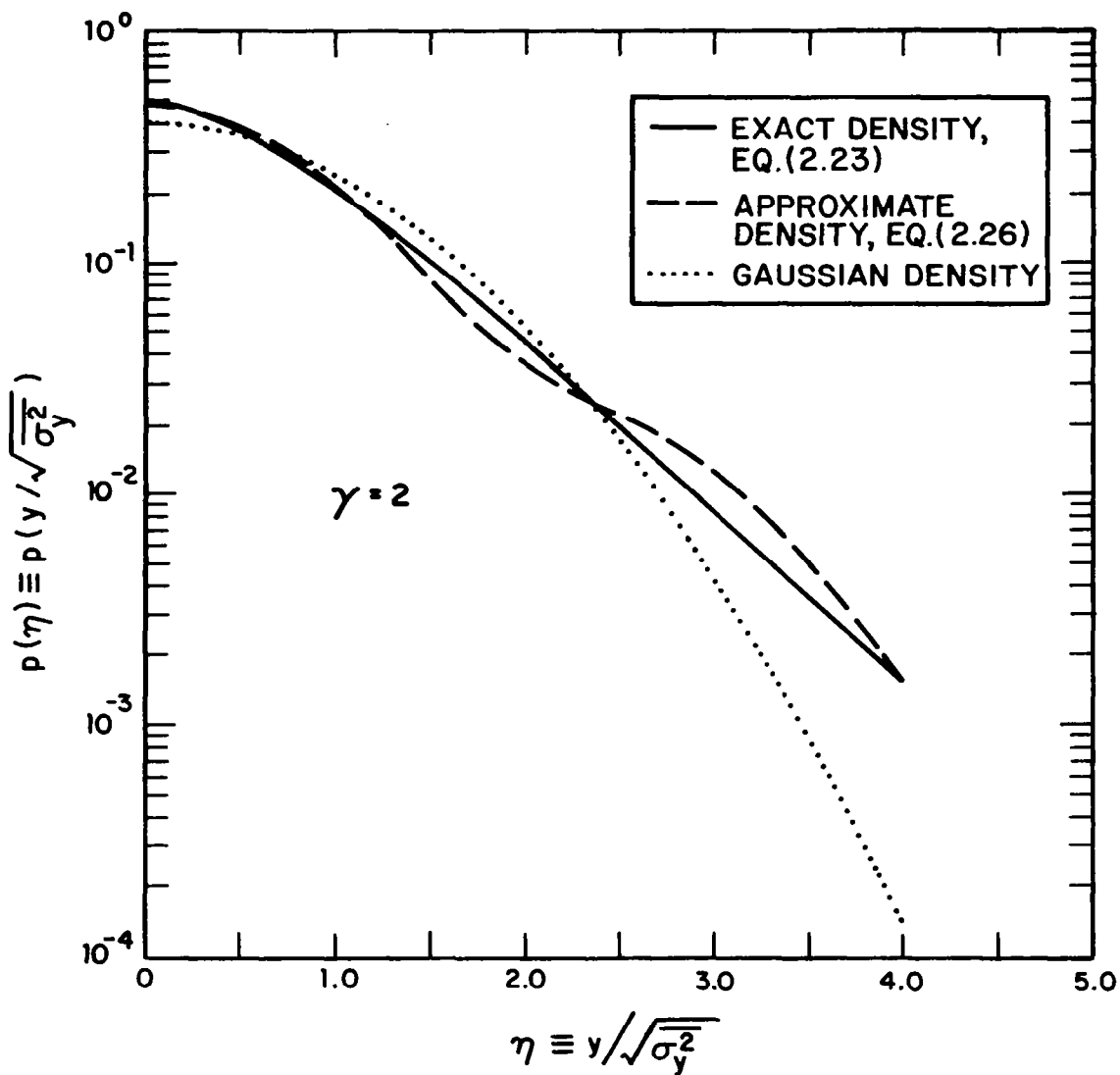


FIG. 14. COMPARISON OF EXACT AND APPROXIMATE PROBABILITY DENSITY FUNCTIONS, EQS. (2.23) AND (2.26) RESPECTIVELY, FOR $\gamma = 2$. DEFINITION OF γ IS GIVEN IN FIG. 8.

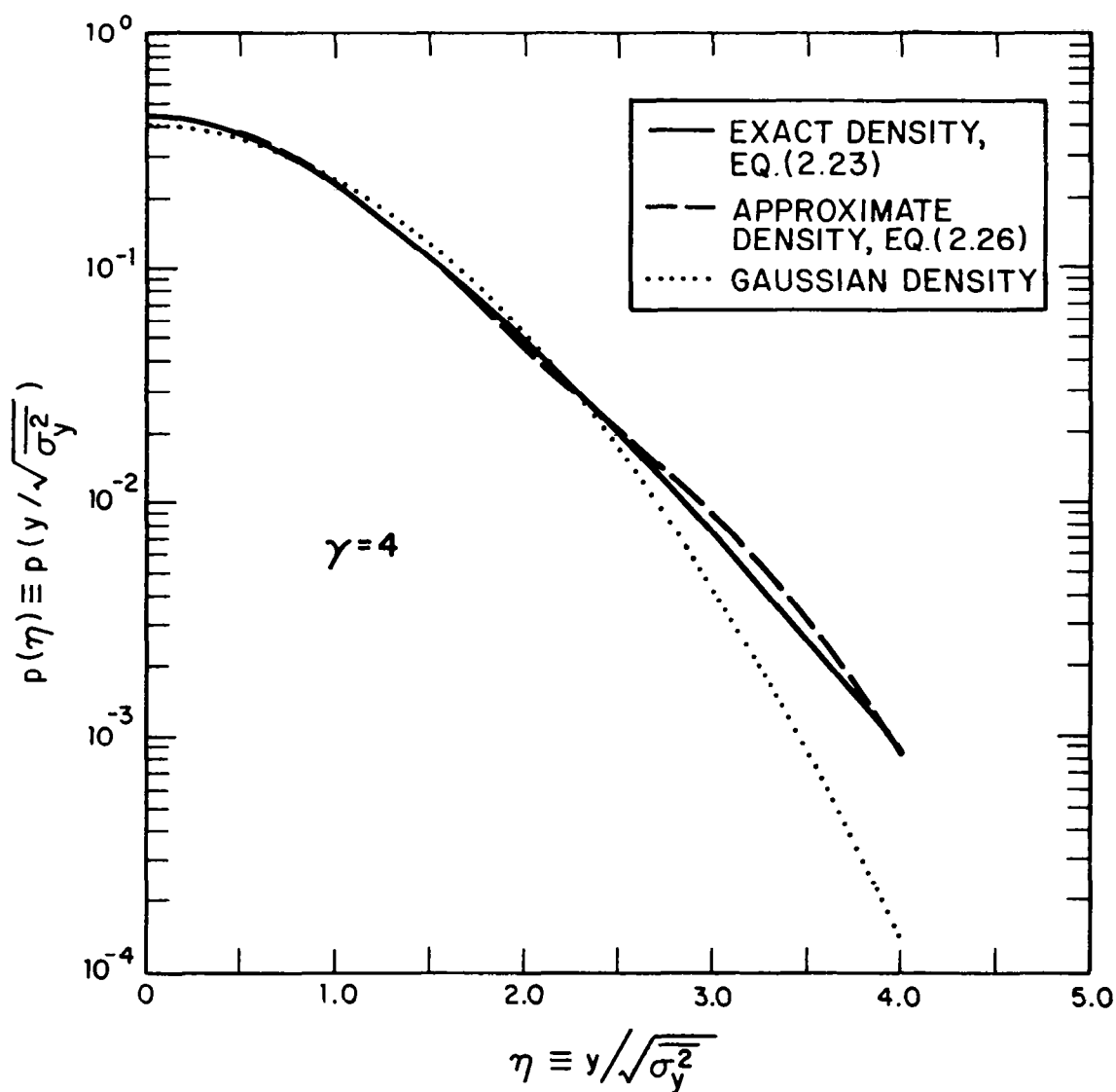


FIG. 15. COMPARISON OF EXACT AND APPROXIMATE PROBABILITY DENSITY FUNCTIONS, EQS. (2.23) AND (2.26) RESPECTIVELY, FOR $\gamma = 4$. DEFINITION OF γ IS GIVEN IN FIG. 8.

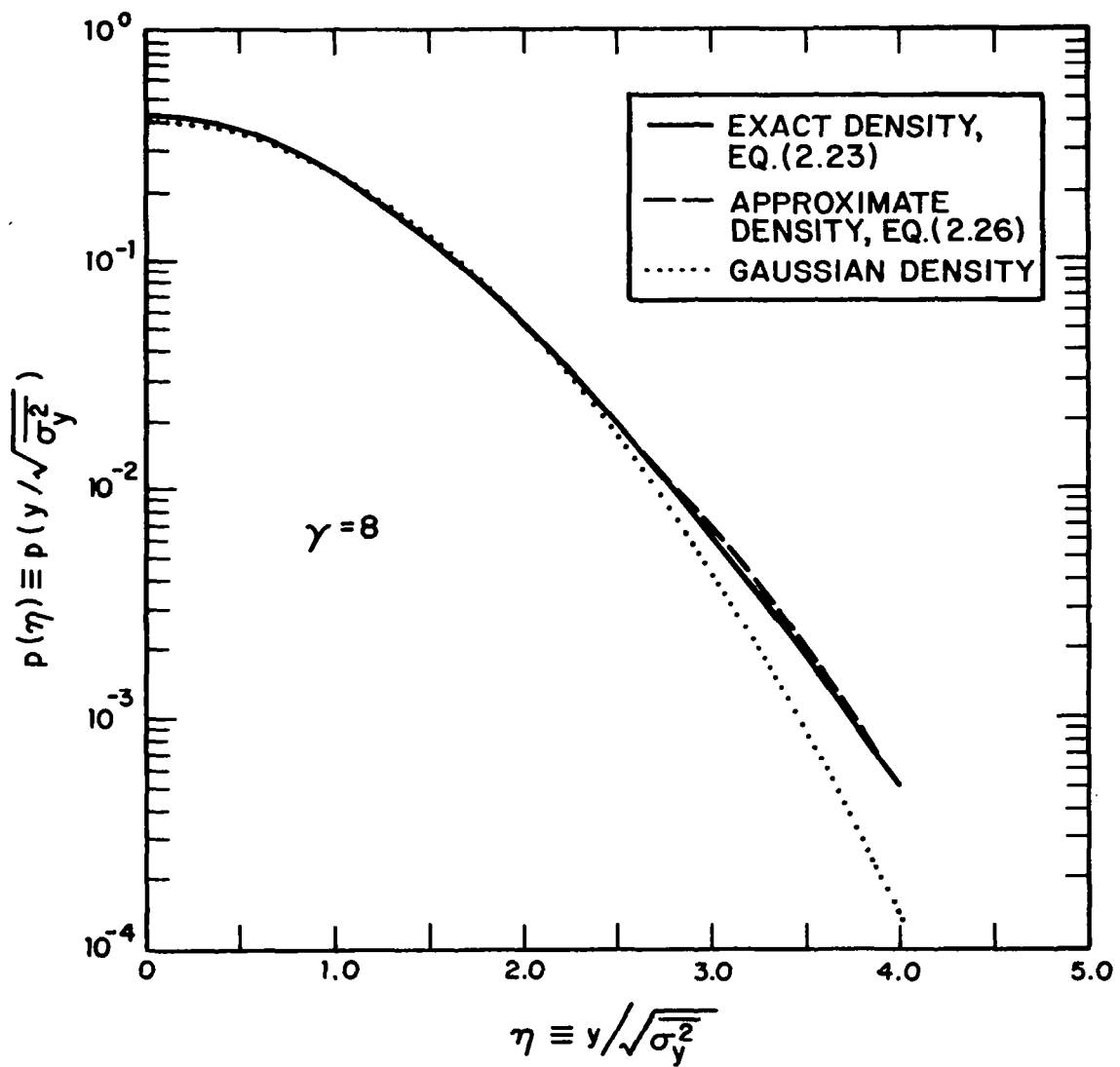


FIG. 16. COMPARISON OF EXACT AND APPROXIMATE PROBABILITY DENSITY FUNCTIONS, EQS. (2.23) AND (2.26) RESPECTIVELY, FOR $\gamma = 8$. DEFINITION OF γ IS GIVEN IN FIG. 8.

depends only on properties of the turbulence. Thus, the results to follow will be valid for aircraft impulse response functions of arbitrary long duration — and, therefore, for supersonic as well as subsonic aircraft. First, we shall obtain a general expression for the mean-square aircraft response.

General expression for mean-square aircraft response. Equation (1.11) is the general input-response relationship for spectra, where $\Phi_W(f)$ is the two-sided power spectral density of the turbulence, $H(f)$ is the aircraft complex frequency response function, and $\Phi_y(f)$ is the aircraft response power spectral density. The aircraft mean-square response is obtained by integrating Eq. (1.11) over $-\infty < f < \infty$:

$$E\{y^2\} \equiv \overline{\sigma_y^2} = \int_{-\infty}^{\infty} \Phi_W(f) |H(f)|^2 df . \quad (2.28)$$

The bar over σ_y^2 is required since $E\{y^2\} = \overline{\sigma_y^2}$ is the unconditioned expected value of Eq. (2.4).

From the mutual statistical independence of the processes $\{w_s(t)\}$, $\{\sigma_f(t)\}$, and $\{z(t)\}$ in our model of Eq. (1.2), it follows that the processes $\{w_s(t)\}$ and $\{w_f(t)\}$ are independent. Therefore, the power spectrum of $\{w(t)\}$ is the sum of the power spectra of $\{w_s(t)\}$ and $\{w_f(t)\}$ — i.e.,

$$\Phi_W(f) = \Phi_{w_s}(f) + \Phi_{w_f}(f) . \quad (2.29)$$

Substitution of Eq. (2.29) into Eq. (2.28) shows that the mean-square aircraft response is the sum of the mean-square responses from the slow and fast turbulence components — i.e.,

$$E\{y^2\} \equiv \overline{\sigma_y^2} = \sigma_{y_s}^2 + \overline{\sigma_{y_f}^2} , \quad (2.30)$$

where

$$\sigma_{y_s}^2 = \int_{-\infty}^{\infty} \Phi_{w_s}(f) |H(f)|^2 df , \quad (2.31)$$

and

$$\overline{\sigma_{y_f}^2} = E\{\sigma_{y_f}^2(t)\}, \quad (2.32)$$

where $\sigma_{y_f}^2(t)$ is the conditional mean-square aircraft response to the fast turbulence component conditioned on the random function $\sigma_f(t)$ as in Eq. (2.4) - i.e.,

$$\sigma_{y_f}^2 = \sigma_{y_f}^2(t) \triangleq E\{y_f^2(t) | \sigma_f(u)\}, \quad -\infty < u \leq t. \quad (2.33)$$

General expression for second central moment of conditional instantaneous mean-square response. Since the slow turbulence component $w_s(t)$ is independent of $\sigma_f(t)$, it follows from Eq. (2.4) that the conditional mean-square response $\sigma_{y_f}^2$ may be expressed as

$$\sigma_y^2 = \sigma_{y_s}^2 + \sigma_{y_f}^2, \quad (2.34)$$

where $\sigma_{y_s}^2$ is the unconditioned mean-square response to the slow component $w_s(t)$ given by Eq. (2.31), and $\sigma_{y_f}^2$ is the conditional mean-square response to the fast component defined by Eq. (2.33). The second central moment of σ_y^2 - defined by Eq. (2.11) for $k = 2$ - therefore can be expressed as

$$\mu_{\sigma_y^2}^{(2)} \triangleq E\{(\sigma_y^2 - \overline{\sigma_y^2})^2\} \quad (2.35a)$$

$$= E\{[(\sigma_{y_s}^2 + \sigma_{y_f}^2) - (\sigma_{y_s}^2 + \overline{\sigma_{y_f}^2})]^2\} \quad (2.35b)$$

$$= E\{(\sigma_{y_f}^2 - \overline{\sigma_{y_f}^2})^2\}, \quad (2.35c)$$

where we have introduced Eq. (2.34) in the second line and have used the fact that $\sigma_{y_s}^2$ is not a random variable. We see from Eq. (2.35c) that $\mu_{\sigma_y^2}^{(2)}$ depends only on the aircraft response to the *fast* turbulence component.

Furthermore, we see from Eq. (2.24) that the overall coefficient of the correction term to the Gaussian approximation of the response density function is proportional to

$$\frac{\mu_{\sigma_y}^{(2)}}{\sigma_y^2} = \frac{E\{(\sigma_{y_f}^2 - \overline{\sigma_{y_f}^2})^2\}}{\sigma_{y_s}^2 + \overline{\sigma_{y_f}^2}}, \quad (2.36)$$

according to Eqs. (2.30) and (2.35c). Hence, $\mu_{\sigma_y}^{(2)}/\overline{\sigma_y^2}$ decreases as the mean-square response $\sigma_{y_s}^2$ to the slow turbulence component is increased. This behavior is in agreement with intuition, since the response to the slow component $w_s(t)$ is assumed to be Gaussian; therefore, the magnitude of the correction to the Gaussian response term decreases as the fraction of the response in the Gaussian component increases.

Series expansion of conditional instantaneous mean-square response. A series of developments in Refs. 34, 18, and 19 has lead to a series expansion for the conditional instantaneous power spectrum of the "fast" component $w_f(t) = \sigma_f(t) z(t)$ of our turbulence model. This expansion is given on p. 23 of Ref. 19 as

$$\Phi_{w_f}(f, t | \sigma_f) = \sum_{n=0}^N \frac{a_n(t)}{n!} \Phi_z^{(n)}(f) + R_{N+1}(f, t), \quad (2.37)$$

where $\Phi_z^{(n)}(f)$ is defined as the n th derivative of the power spectral density $\Phi_z(f)$ of $\{z(t)\}$ - i.e.,

$$\Phi_z^{(n)}(f) \triangleq \frac{d^n}{df^n} \Phi_z(f) \quad (2.38)$$

- and where

$$\Phi_z^{(0)}(f) \equiv \Phi_z(f) \quad (2.39)$$

The coefficients $a_n(t)$ in Eq. (2.37) may be expressed in terms of the derivatives of $\sigma_f(t)$ by

$$a_n(t) = \frac{1}{(-i4\pi)^n} \sum_{k=0}^n (-1)^k \binom{n}{k} \frac{d^k \sigma_f(t)}{dt^k} \frac{d^{n-k} \sigma_f(t)}{dt^{n-k}}, \quad (2.40)$$

where

$$\binom{n}{k} = \frac{n!}{(n-k)!k!} \quad (2.41)$$

are the binomial coefficients, and where we remind the reader that the conditioning on the random intensity $\sigma_f(t)$ indicated by the left-hand side of Eq. (2.37) implies that $\sigma_f(t)$ in Eq. (2.40) and the $a_n(t)$ are to be treated as known functions. From Eq. (2.40), one may show that for odd integer values of n , we have

$$a_n(t) = 0, \quad n = \text{odd}. \quad (2.42)$$

Expressions for the remainder term $R_{N+1}(f,t)$ in Eq. (2.37) are given by Eqs. (4.7) and (4.13) on pp. 27 and 29 of Ref. 18. The first two nonvanishing terms $a_n(t)$ can be expressed as

$$a_0(t) = \sigma_f^2(t) \quad (2.43)$$

and

$$a_2(t) = - \frac{1}{8\pi^2} \sigma_f^2(t) \frac{d^2 \ln \sigma_f(t)}{dt^2} \quad (2.44a)$$

$$= - \frac{1}{16\pi^2} \sigma_f^2(t) \frac{d^2 \ln[\sigma_f^2(t)]}{dt^2}. \quad (2.44b)$$

Hence, we may express the first two nonvanishing terms of the series in Eq. (2.37) as

$$\Phi_{w_f}(f, t | \sigma_f) = \sigma_f^2(t) \left[\Phi_z(f) - \frac{1}{32\pi^2} \frac{d^2 \ln \sigma_f^2(t)}{dt^2} \Phi_z^{(2)}(f) \right] + \dots \quad (2.45)$$

According to p. 26 of Ref. 19, we can express the conditional instantaneous spectrum of the aircraft response to the "fast" turbulence component $w_f(t)$ as

$$\Phi_{y_f}(f, t | \sigma_f) = \int_{-\infty}^{\infty} \Phi_{w_f}(f, t - \tau | \sigma_f) \Phi_h(f, \tau) d\tau, \quad (2.46)$$

where $\Phi_h(f, t)$ is the instantaneous power spectrum [34] of the aircraft unit-impulse response of interest:

$$\Phi_h(f, t) = \int_{-\infty}^{\infty} h(t - \frac{\tau}{2}) h(t + \frac{\tau}{2}) e^{-i2\pi f \tau} d\tau. \quad (2.47)$$

Substitution of Eq. (2.46) into Eq. (2.37) gives — ignoring the remainder term —

$$\Phi_{y_f}(f, t | \sigma_f) = \sum_{n=0}^N \frac{1}{n!} \int_{-\infty}^{\infty} a_n(t - \tau) \Phi_z^{(n)}(f) \Phi_h(f, \tau) d\tau. \quad (2.48)$$

To obtain an expression for the conditional instantaneous mean-square response to the fast component $w_f(t)$ as in Eq. (2.33), we integrate $\Phi_{y_f}(f, t | \sigma_f)$ over all f :

$$\begin{aligned} \sigma_{y_f}^2 &\equiv \sigma_{y_f}^2(t) \triangleq E\{y_f^2(t) | \sigma_f(u)\}, \quad -\infty < u \leq t \\ &= \int_{-\infty}^{\infty} \Phi_{y_f}(f, t | \sigma_f) df \end{aligned}$$

or

$$\begin{aligned}
\sigma_{y_f}^2 &\equiv \sigma_{y_f}^2(t) = \sum_{n=0}^N \frac{1}{n!} \int_{-\infty}^{\infty} a_n(t-\tau) \int_{-\infty}^{\infty} \phi_z^{(n)}(f) \phi_h(f, \tau) df d\tau \\
&= \sum_{n=0}^N \frac{1}{n!} \int_{-\infty}^{\infty} a_n(t-\tau) \gamma_{h,z,n}(\tau) d\tau, \quad (2.49)
\end{aligned}$$

where, in going to the third line, we have used Eq. (2.48) and have interchanged the order of integration, and in going to the last line, we have used the definition

$$\gamma_{h,z,n}(t) \triangleq \int_{-\infty}^{\infty} \phi_z^{(n)}(f) \phi_h(f, t) df, \quad (2.50)$$

which is a function of the n th derivative of the power spectral density $\phi_z(f)$ of the turbulence component $\{z(t)\}$ and the instantaneous spectral density $\phi_h(f, t)$ of the aircraft unit-impulse response function $h(t)$. The result of Eq. (2.49) ignores the contribution of the remainder term $R_{N+1}(f, t)$ in Eq. (2.37). Equation (2.49) is the desired series expansion for the conditional instantaneous mean-square aircraft response to the "fast" turbulence component $w_f(t) = \sigma_f(t) z(t)$.

Series expansion of second moment of conditional instantaneous mean-square response. An expression for the coefficient $\mu_{\sigma_y^2}^{(2)} / (\sigma_y^2)^2$ of the correction term for the Gaussian probability density in Eq. (2.24) is given by Eq. (2.36), where we see that the second central moment of $\sigma_{y_f}^2$ is required. We may obtain a series expansion for the second moment of $\sigma_{y_f}^2$ by taking the expected value of the square of Eq. (2.49):

$$\begin{aligned}
E\{(\sigma_{y_f}^2)^2\} &= \sum_{n_1=0}^N \sum_{n_2=0}^N \frac{1}{n_1! n_2!} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} E\{a_{n_1}(t'-\tau_1) a_{n_2}(t'-\tau_2)\} \\
&\quad \times \gamma_{h,z,n_1}(\tau_1) \gamma_{h,z,n_2}(\tau_2) d\tau_1 d\tau_2, \quad (2.51)
\end{aligned}$$

where the expectation is taken with respect to fluctuations in the random function $\sigma_f(t)$, and where t has been replaced by t' . We are primarily interested in situations where the random process $\{\sigma_f(t)\}$ is stationary. In these cases, we may take $t = t' - \tau_1$; hence,

$$E\{a_{n_1}(t' - \tau_1) a_{n_2}(t' - \tau_2)\} = E\{a_{n_1}(t) a_{n_2}(t + \tau_1 - \tau_2)\} \quad (2.52)$$

which is independent of t .

Introduction of locally stationary assumption. Our main interest in these developments is situations where the locally stationary assumption of Eq. (1.8) is valid. From Eqs. (1.8a) and (2.43) through (2.45), we see that the locally stationary assumption permits us to include only the term $n = 0$ in Eq. (2.49) — i.e.,

$$\begin{aligned} \sigma_{y_f}^2(t) &= E\{y_f^2(t) | \sigma_f(u)\} \quad -\infty < u \leq t \\ &= \int_{-\infty}^{\infty} \sigma_f^2(t - \tau) \gamma_{h,z}(\tau) d\tau, \end{aligned} \quad (2.53)$$

where we have used Eq. (2.43) and the notation

$$\gamma_{h,z}(t) \equiv \gamma_{h,z,0}(t) = \int_{-\infty}^{\infty} \phi_z(f) \phi_h(f, t) df \quad (2.54)$$

which follows from Eq. (2.50). The expected value of $\sigma_{y_f}^2$ [with respect to fluctuations in $\sigma_f(t)$] is

$$\begin{aligned} \overline{\sigma_{y_f}^2} &\equiv E\{\sigma_{y_f}^2(t)\} \\ &= \int_{-\infty}^{\infty} \overline{\sigma_f^2} \gamma_{h,z}(\tau) d\tau; \end{aligned}$$

hence, substituting the mean value of $\sigma_{y_f}^2(t)$, we have from Eqs. (2.53) and the above relationship,

$$\sigma_{y_f}^2(t) - \overline{\sigma_{y_f}^2} = \int_{-\infty}^{\infty} [\sigma_f^2(t-\tau) - \overline{\sigma_f^2}] \gamma_{h,z}(\tau) d\tau. \quad (2.55)$$

According to Eqs. (2.24) and (2.35c) we require the second central moment $\mu_{\sigma_y^2}^{(2)}$ which is given by the mean-square value of Eq. (2.55). However, the right-hand side of Eq. (2.55) is the convolution of $[\sigma_f^2(t) - \overline{\sigma_f^2}]$ and $\gamma_{h,z}(t)$ — i.e., Eq. (2.55) may be thought of as describing the response of a linear two-terminal time-invariant system with unit-impulse response $\gamma_{h,z}(t)$ to the stochastic input $[\sigma_f^2(t) - \overline{\sigma_f^2}]$. Thus, we may express the power spectral density $\Phi_{\sigma_{y_f}^2}(f)$ of the process $\{\sigma_{y_f}^2(t) - \overline{\sigma_{y_f}^2}\}$ as the product of the power spectral density $\Phi_{\sigma_f^2}(f)$ of $\{\sigma_f^2(t) - \overline{\sigma_f^2}\}$ and $|\tilde{\gamma}_{h,z}(f)|^2$, where

$$\tilde{\gamma}_{h,z}(v) \triangleq \int_{-\infty}^{\infty} \gamma_{h,z}(t) e^{i2\pi vt} dt \quad (2.56)$$

is the (inverse) Fourier transform of the "system impulse response" $\gamma_{h,z}(t)$:

$$\Phi_{\sigma_{y_f}^2}(f) = \Phi_{\sigma_f^2}(f) |\tilde{\gamma}_{h,z}(f)|^2. \quad (2.57)$$

The mean-square value of $\{\sigma_{y_f}^2(t) - \overline{\sigma_{y_f}^2}\}$ which, according to Eq. (2.35c) is the second central moment $\mu_{\sigma_y^2}^{(2)}$ is obtained by integrating Eq. (2.57) over all f :

$$\mu_{\sigma_y^2}^{(2)} = \int_{-\infty}^{\infty} \Phi_{\sigma_f^2}(f) |\tilde{\gamma}_{h,z}(f)|^2 df. \quad (2.58)$$

Equations (2.28) and (2.58) are the required relationships for evaluation of the coefficient $\mu_{\sigma_y^2}^{(2)}/(\overline{\sigma_y^2})^2$ of the correction term that occurs in Eq. (2.24) to the Gaussian probability density function of the aircraft response $y(t)$. The turbulence characterizations required for evaluation of $\overline{\sigma_y^2}$ and $\mu_{\sigma_y^2}^{(2)}$ are the power spectral density $\Phi_w(f)$ of the turbulence velocity $\{w(t)\}$ and the power spectral density $\Phi_{\sigma_f^2}(f)$ of $\{\sigma_f^2(t) - \overline{\sigma_f^2}\}$, where $\sigma_f(t)$ is the intensity modulation f in the fast component $w_f(t)$ given by Eq. (1.3). Methods for computing $\Phi_{\sigma_f^2}(f)$ from turbulence recordings are described on pp. 79-83 of Ref. 19. The quantity $\tilde{\gamma}_{h,z}(f)$ is defined by Eqs. (2.54) and (2.56). Notice that $\tilde{\gamma}_{h,z}(f)$ depends both on the power spectral density $\Phi_z(f)$ of the turbulence component $\{z(t)\}$ and on the aircraft unit-impulse response function as may be seen from Eq. (2.47). Thus, $\tilde{\gamma}_{h,z}(f)$ or $\gamma_{h,z}(t)$ characterizes the aircraft impulse response with respect to the turbulence component $\{z(t)\}$. The locally stationary assumption of Eq. (1.8) has been used in deriving Eq. (2.58).

Alternative form for system characterization. Generalizing the definition of Eq. (2.56) to include $\gamma_{h,z,n}(t)$ - which is defined by Eq. (2.50) - we have

$$\begin{aligned}
 \tilde{\gamma}_{h,z,n}(v) &\triangleq \int_{-\infty}^{\infty} \gamma_{h,z,n}(t) e^{i2\pi vt} dt \\
 &= \int_{-\infty}^{\infty} \left[\int_{-\infty}^{\infty} \Phi_z^{(n)}(f) \Phi_h(f,t) df \right] e^{i2\pi vt} dt \\
 &= \int_{-\infty}^{\infty} \Phi_z^{(n)}(f) \left[\int_{-\infty}^{\infty} \Phi_h(f,t) e^{i2\pi vt} dt \right] df . \quad (2.59)
 \end{aligned}$$

However, using Eq. (35b) on p. 29 of Ref. 34:

$$\begin{aligned}\tilde{\Phi}_h(f, \nu) &\triangleq \int_{-\infty}^{\infty} \Phi_h(f, t) e^{i2\pi \nu t} dt \\ &= H(f - \frac{\nu}{2}) H^*(f + \frac{\nu}{2}) ,\end{aligned}\tag{2.60}$$

where $H(f)$ is the complex frequency response defined by Eq. (1.9), we have from Eqs. (2.59) and (2.60)

$$\tilde{\gamma}_{h,z,n}(\nu) = \int_{-\infty}^{\infty} \Phi_z^{(n)}(f) H(f - \frac{\nu}{2}) H^*(f + \frac{\nu}{2}) df.\tag{2.61}$$

The special case of Eq. (2.61) for use in Eq. (2.58) is

$$\tilde{\gamma}_{h,z}(\nu) = \int_{-\infty}^{\infty} \Phi_z(f) H(f - \frac{\nu}{2}) H^*(f + \frac{\nu}{2}) df ,\tag{2.62}$$

which is the desired alternative form of $\tilde{\gamma}_{h,z}(\nu)$ for use in Eq. (2.58). Equation (2.62) may be interpreted with the aid of the material contained in Sec. 4 of Ref. 34.

Limiting Cases of Expansion Coefficients of the NonGaussian Term

Limiting cases of the central result, Eq. (2.58), provide insight into that result. Three limiting cases are discussed below.

Case 1: $w_f(t)$ is stationary and Gaussian. Since the process $\{z(t)\}$ in our model of Eq. (1.3) is stationary and Gaussian, the "fast" turbulence component $\{w_f(t)\}$ is stationary and Gaussian when

$$\sigma_f(t) = \text{constant} = \overline{\sigma_f} .\tag{2.63}$$

In this case, the power spectrum $\Phi_{\sigma_f^2}(f)$ of the random process $\{\sigma_f^2(t) - \overline{\sigma_f^2}\}$ is zero. Therefore, according to Eq. (2.58),

$\mu_{\sigma_y^2}^{(2)}$ is zero, and according to Eq. (2.24), the probability density of the aircraft response $y(t)$ is Gaussian.

Case 2: power spectrum of $\{z(t)\}$ is white. This case is of little direct interest for the representation of turbulence since $\{z(t)\}$ would normally be taken to have the appropriate (transverse or longitudinal) von Karman spectral form. Nevertheless, it is of general interest as a limiting case in the study of Eq. (2.58). For this case, we have

$$\Phi_z(f) = \text{constant} = \Phi_z(\cdot) \quad . \quad (2.64)$$

Using Eq. (2.64), we have from Eq. (2.54):

$$\begin{aligned} \gamma_{h,z}(t) &= \Phi_z(\cdot) \int_{-\infty}^{\infty} \Phi_h(f,t) df \\ &= \Phi_z(\cdot) h^2(t) \quad , \end{aligned} \quad (2.65)$$

where, in going to the second line, we have used Eq. (12a) on p. 26 of Ref. 34 applied to the aircraft unit-impulse response $h(t)$. Therefore, according to Eq. (2.56), we have for the present case

$$\tilde{\gamma}_{h,z}(\nu) = \Phi_z(\cdot) \int_{-\infty}^{\infty} h^2(t) e^{i2\pi\nu t} dt \quad , \quad (2.66)$$

which is the (inverse) Fourier transform of $h^2(t)$ multiplied by the constant spectral density $\Phi_z(\cdot)$.

Equation (2.66) may be used directly in Eq. (2.58) to evaluate $\mu_{\sigma_y^2}^{(2)}$. Alternatively, we may define the "autocorrelation function" of the square of the (deterministic) unit-impulse response $h(t)$ as

$$\phi_{h^2}(\tau) \triangleq \int_{-\infty}^{\infty} h^2(t) h^2(t+\tau) dt \quad , \quad (2.67)$$

which is necessarily an even function of τ . Using Weiner's theorem — e.g., Eq. (135) on p. 54 of Ref. 34 — we then have from Eqs. (2.66) and (2.67),

$$|\tilde{\gamma}_{h,z}(\nu)|^2 = \Phi_z^2(\cdot) \int_{-\infty}^{\infty} \phi_{h^2}(\tau) e^{i2\pi\nu\tau} d\tau, \quad (2.68)$$

which may be used to evaluate $\mu_{\sigma_y^2}^{(2)}$ by Eq. (2.58). Notice that $h^2(t)$ is a nonnegative function. Therefore, for reasonably well behaved $h(t)$, the main contribution to $\tilde{\gamma}_{h,z}(\nu)$ will be in the region about the origin $\nu = 0$, and the nominal bandwidth of $|\tilde{\gamma}_{h,z}(\nu)|^2$ will be of the order of the reciprocal of the nominal duration of $h(t)$. Therefore, this same low-frequency region of the spectrum $\Phi_{\sigma_f^2}(\nu)$ is relevant in the computation of $\mu_{\sigma_y^2}^{(2)}$ by Eq. (2.58).

Case 3: fluctuations in $\sigma_f(t)$ are negligible over the duration of $h(t)$. When fluctuations in the intensity modulation $\sigma_f(t)$ occur sufficiently slowly, the power spectral density of $\{\sigma_f^2(t) - \overline{\sigma_f^2}\}$ has all of its area concentrated near zero frequency. The limiting case in this situation is

$$\Phi_{\sigma_f^2}(\nu) = E\{(\sigma_f^2 - \overline{\sigma_f^2})^2\} \delta(\nu), \quad (2.69)$$

where $\delta(\nu)$ is the Dirac delta function located at $\nu = 0$. Substitution of Eq. (2.69) into Eq. (2.58) yields

$$\begin{aligned} \mu_{\sigma_y^2}^{(2)} &= E\{(\sigma_f^2 - \overline{\sigma_f^2})^2\} \int_{-\infty}^{\infty} \delta(\nu) |\tilde{\gamma}_{h,z}(\nu)|^2 d\nu \\ &= E\{(\sigma_f^2 - \overline{\sigma_f^2})^2\} |\tilde{\gamma}_{h,z}(0)|^2. \end{aligned} \quad (2.70)$$

However, from Eq. (2.62) it follows directly that

$$\tilde{\gamma}_{h,z}(0) = \int_{-\infty}^{\infty} \Phi_z(f) |H(f)|^2 df, \quad (2.71)$$

which is real; hence, Eqs. (2.70) and (2.71) yield

$$\mu_{\sigma_y^2}^{(2)} = E\{(\sigma_f - \overline{\sigma_f^2})^2\} \left[\int_{-\infty}^{\infty} \Phi_z(f) |H(f)|^2 df \right]^2, \quad (2.72)$$

which is the desired result.

From Eq. (2.71), we recognize $\tilde{y}_{h,z}(0)$ as the mean-square response of the aircraft to the turbulence component $\{z(t)\}$. When Eq. (2.72) is combined with Eq. (2.24), we see that the resulting expression for the probability density function of the aircraft response is identical with that given by Eqs. (4.50) and (4.51) on p. 48 of Ref. 19. These results were derived under the assumptions that (i) fluctuations in $\sigma_f(t)$ occur slowly in comparison with those of $z(t)$, and (ii) variations in $\sigma_f(t)$ are negligible over durations comparable with the aircraft impulse response duration. The more general result of Eq. (2.58) relaxes, completely, the requirements for assumption (ii). However, assumption (i) — which is described by Eq. (1.8) of this report — has been used in obtaining Eq. (2.58). We again emphasize that the assumption of Eq. (1.8) depends on turbulence properties alone and is believed to be generally satisfied by atmospheric turbulence.

Series Representation of Expansion Coefficient of the NonGaussian Term

The limiting case of Eqs. (2.70) — (2.72) suggests a series expansion for $\mu_{\sigma_y^2}^{(2)}$ that may be useful in situations where assumption (ii) above is almost or only marginally satisfied. Equation (2.70) depends on the frequency content of $|\tilde{y}_{h,z}(v)|^2$ only at $v = 0$. This fact suggests a series representation of $\mu_{\sigma_y^2}^{(2)}$ obtained by first expanding $|\tilde{y}_{h,z}(v)|^2$ in a Maclaurin series and then integrating term by term.

The Maclaurin expansion of $|\tilde{y}_{h,z}(v)|^2$ may be expressed as

$$|\tilde{y}_{h,z}(v)|^2 = \sum_{n=0}^{\infty} \frac{v^n}{n!} \frac{d^n}{dv^n} \left[|\tilde{y}_{h,z}(v)|^2 \right]_{v=0}. \quad (2.73)$$

By Leibniz's rule for the n th derivative of a product (p. 111 of Ref. 35), we have

$$\begin{aligned} \frac{d^n}{dv^n} |\tilde{\gamma}_{h,z}(v)|^2 &= \frac{d^n}{dv^n} \left[\tilde{\gamma}_{h,z}^*(v) \tilde{\gamma}_{h,z}(v) \right] \\ &= \sum_{k=0}^n \binom{n}{k} \frac{d^k \tilde{\gamma}_{h,z}^*(v)}{dv^k} \frac{d^{n-k} \tilde{\gamma}_{h,z}(v)}{dv^{n-k}}. \end{aligned} \quad (2.74)$$

However, from Eq. (2.62), we have

$$\begin{aligned} \left. \frac{d^k \tilde{\gamma}_{h,z}(v)}{dv^k} \right|_{v=0} &= \int_{-\infty}^{\infty} \Phi_z(f) \frac{\partial^k}{\partial v^k} \left[H\left(f - \frac{v}{2}\right) H^*\left(f + \frac{v}{2}\right) \right] \bigg|_{v=0} df \\ &= (12\pi)^k \int_{-\infty}^{\infty} \Phi_z(f) m_h^{(k)}(f) df, \end{aligned} \quad (2.75)$$

where we have used Eq. (79) on p. 269 of Ref. 36 as applied to the deterministic (complex) function $H(f)$, and where $m_h^{(k)}(f)$ is the k th power-moment spectrum of the aircraft unit-impulse response function $h(t)$. Properties of power-moments spectra are discussed on pp. 264-269 and 281-288 of Ref. 36. Their name arises from the fact that their integrals over $-\infty < f < \infty$ satisfy

$$\int_{-\infty}^{\infty} m_h^{(k)}(f) df = \int_{-\infty}^{\infty} t^k h^2(t) dt, \quad k=0,1,2,\dots, \quad (2.76)$$

where the right-hand side of Eq. (2.76) may be interpreted as the k th time-moment of the instantaneous "power" $h^2(t)$ of $h(t)$. Furthermore, the power-moments spectra $m_h^{(k)}(f)$ are real and even functions of the frequency variable f . From Eq. (2.75), we also have

$$\left. \frac{d^k \tilde{\gamma}_{h,z}^*(v)}{dv^k} \right|_{v=0} = (-i2\pi)^k \int_{-\infty}^{\infty} \Phi_z(f) m_h^{(k)}(f) df . \quad (2.77)$$

Let us define

$$\Gamma_{h,z}^{(k)} \triangleq \int_{-\infty}^{\infty} \Phi_z(f) m_h^{(k)}(f) df . \quad (2.78)$$

Then combining Eqs. (2.74), (2.76), (2.77), and (2.78), it follows that

$$\frac{d^n}{dv^n} \left[|\tilde{\gamma}_{h,z}(v)|^2 \right]_{v=0} = (i2\pi)^n \sum_{k=0}^n (-1)^k \binom{n}{k} \Gamma_{h,z}^{(k)} \Gamma_{h,z}^{(n-k)} . \quad (2.79)$$

Hence, from Eqs. (2.58), (2.73), and (2.79), we have

$$\begin{aligned} \mu_{\sigma_y^2}^{(2)} &= \sum_{n=0}^{\infty} \frac{1}{n!} \int_{-\infty}^{\infty} (i2\pi v)^n \Phi_{\sigma_f^2}(v) dv \\ &\times \sum_{k=0}^n (-1)^k \binom{n}{k} \Gamma_{h,z}^{(k)} \Gamma_{h,z}^{(n-k)} . \end{aligned} \quad (2.80)$$

However, the autocorrelation function $\phi_{\sigma_f^2}(t)$ of the random process $\{\sigma_f^2(t) - \overline{\sigma_f^2}\}$ is the inverse Fourier transform of the power spectrum $\Phi_{\sigma_f^2}(v)$ - i.e.,

$$\phi_{\sigma_f^2}(t) = \int_{-\infty}^{\infty} \Phi_{\sigma_f^2}(v) e^{i2\pi vt} dv . \quad (2.81)$$

Differentiating both sides of Eq. (2.81) n times with respect to t yields

$$\begin{aligned}\phi_{\sigma_f^2}^{(n)}(t) &\triangleq \frac{d^n}{dt^n} \phi_{\sigma_f^2}(t) \\ &= \int_{-\infty}^{\infty} (i2\pi v)^n \phi_{\sigma_f^2}(v) e^{i2\pi v t} dv;\end{aligned}\quad (2.82)$$

hence,

$$\phi_{\sigma_f^2}^{(n)}(0) = \int_{-\infty}^{\infty} (i2\pi v)^n \phi_{\sigma_f^2}(v) dv. \quad (2.83)$$

Let us define

$$s_{h,z}^{(n)} \triangleq \sum_{k=0}^n (-1)^k \binom{n}{k} \Gamma_{h,z}^{(k)} \Gamma_{h,z}^{(n-k)}. \quad (2.84)$$

Combining Eqs. (2.80), (2.83), and (2.84) we have, finally,

$$\mu_{\sigma_y^2}^{(2)} = \sum_{n=0}^{\infty} \frac{1}{n!} \phi_{\sigma_f^2}^{(n)}(0) s_{h,z}^{(n)}, \quad (2.85)$$

which is the desired series expansion of $\mu_{\sigma_y^2}^{(2)}$.

The power-moments spectra are defined on p. 260 of Ref. 36 as

$$m_h^{(k)}(f) \triangleq \int_{-\infty}^{\infty} t^k \phi_h(f, t) dt, \quad k=0,1,2,\dots \quad (2.86)$$

Hence, by forming the k th moment of $\gamma_{h,z}(t)$ and using Eq. (2.54), we see that $\Gamma_{h,z}^{(k)}$ and $\gamma_{h,z}(t)$ are related by

$$\int_{-\infty}^{\infty} t^k \gamma_{h,z}(t) dt = \int_{-\infty}^{\infty} \phi_z(f) \int_{-\infty}^{\infty} t^k \phi_h(f, t) dt df = \Gamma_{h,z}^{(k)}, \quad (2.87)$$

where Eqs. (2.86) and (2.78) have been used in the second equality. According to Eq. (2.87), $\Gamma_{h,z}^{(k)}$ is the k th moment with respect to time of $\gamma_{h,z}(t)$.

Discussion. The terms in Eq. (2.85) for odd values of n are identically zero. To show this, we show that $s_{h,z}^{(n)} = 0$ for $n = \text{odd}$. Consider Eq. (2.84) and let $k' = n-k$; hence, $k = n-k'$. Therefore,

$$\binom{n}{k} = \frac{n!}{(n-k)!k!} = \frac{n!}{(k')!(n-k')!} = \binom{n}{k'} . \quad (2.88)$$

Consequently, we may express the summation in Eq. (2.84) as

$$\begin{aligned} s_{h,z}^{(n)} &= \sum_{k'=0}^n (-1)^{n-k'} \binom{n}{k'} \Gamma_{h,z}^{(n-k')} \Gamma_{h,z}^{(k')} , \\ &= (-1)^n \sum_{k'=0}^n (-1)^{k'} \binom{n}{k'} \Gamma_{h,z}^{(k')} \Gamma_{h,z}^{(n-k')} . \end{aligned} \quad (2.89)$$

Comparison of Eqs. (2.84) and (2.89) shows that

$$s_{h,z}^{(n)} = (-1)^n s_{h,z}^{(n)} , \quad (2.90)$$

from which it follows that

$$s_{h,z}^{(n)} = 0 , \quad n=1,3,5,\dots . \quad (2.91)$$

Therefore, the odd numbered terms in the summation of Eq. (2.85) are identically zero.

It is instructive to consider the first two nonvanishing terms of Eq. (2.85). For $n = 0$, it follows directly from the definition of the autocorrelation function that

$$\phi_{\sigma_f^2}^{(0)}(0) \equiv \phi_{\sigma_f^2}(0) = E\{[\sigma_f^2(t) - \overline{\sigma_f^2}]^2\} . \quad (2.92)$$

Furthermore, one may show [e.g., p. 21 of Ref. 37] that

$$\phi_{\sigma_f^2}^{(2)}(0) = -E \left\{ \left[\frac{d}{dt} \sigma_f^2(t) \right]^2 \right\} , \quad (2.93)$$

which is the negative of the mean-square velocity of $\sigma_f^2(t)$. In addition, we have from Eq. (2.84),

$$s_{h,z}^{(0)} = |\Gamma_{h,z}^{(0)}|^2 \quad (2.94a)$$

$$\begin{aligned} &= \left[\int_{-\infty}^{\infty} \Phi_z(f) \int_{-\infty}^{\infty} \Phi_h(f,t) dt df \right]^2 \\ &= \left[\int_{-\infty}^{\infty} \Phi_z(f) |H(f)|^2 df \right]^2 , \end{aligned} \quad (2.94b)$$

according to Eq. (2.87) above and Eq. (12a) on p. 26 of Ref. 34. Furthermore, from Eq. (2.84) we also have

$$\begin{aligned} s_{h,z}^{(2)} &= \Gamma_{h,z}^{(0)} \Gamma_{h,z}^{(2)} - 2 \left(\Gamma_{h,z}^{(1)} \right)^2 + \Gamma_{h,z}^{(2)} \Gamma_{h,z}^{(0)} \\ &= 2 \left(\Gamma_{h,z}^{(0)} \right)^2 \left[\frac{\Gamma_{h,z}^{(2)}}{\Gamma_{h,z}^{(0)}} - \left(\frac{\Gamma_{h,z}^{(1)}}{\Gamma_{h,z}^{(0)}} \right)^2 \right] . \end{aligned} \quad (2.95)$$

According to Eqs. (2.91), (2.94a), and (2.95), the first non-zero terms of Eq. (2.85) therefore can be expressed as

$$\mu_{\sigma_y^2}^{(2)} \approx \phi_{\sigma_f^2}^{(0)} \left(\Gamma_{h,z}^{(0)} \right)^2 \left\{ 1 + \frac{\phi_{\sigma_f^2}^{(2)}(0)}{\phi_{\sigma_f^2}^{(0)}(0)} \left[\frac{\Gamma_{h,z}^{(2)}}{\Gamma_{h,z}^{(0)}} - \left(\frac{\Gamma_{h,z}^{(1)}}{\Gamma_{h,z}^{(0)}} \right)^2 \right] \right\} . \quad (2.96)$$

From Eqs. (2.92) and (2.94), we see that the first term in Eq. (2.96) is identical to the limiting case of $\mu_{\sigma_y^2}^{(2)}$ given

by Eq. (2.72), which applies to situations where variations in $\sigma_f(t)$ are negligible over durations comparable with the aircraft impulse response duration.

We consider, now, the second term in the right-hand side of Eq. (2.96). Using Eq. (2.87), we see that

$$\begin{aligned} \left[\frac{\Gamma_{h,z}^{(2)}(0)}{\Gamma_{h,z}^{(0)}(0)} - \left(\frac{\Gamma_{h,z}^{(1)}(0)}{\Gamma_{h,z}^{(0)}(0)} \right)^2 \right] &= \frac{\int_{-\infty}^{\infty} t^2 \gamma_{h,z}(t) dt}{\int_{-\infty}^{\infty} \gamma_{h,z}(t) dt} - \left(\frac{\int_{-\infty}^{\infty} t \gamma_{h,z}(t) dt}{\int_{-\infty}^{\infty} \gamma_{h,z}(t) dt} \right)^2 \\ &= \frac{\int_{-\infty}^{\infty} (t - \bar{t})^2 \gamma_{h,z}(t) dt}{\int_{-\infty}^{\infty} \gamma_{h,z}(t) dt}, \end{aligned} \quad (2.97)$$

where

$$\bar{t} \equiv \bar{t}_{h,z} \triangleq \frac{\int_{-\infty}^{\infty} t \gamma_{h,z}(t) dt}{\int_{-\infty}^{\infty} \gamma_{h,z}(t) dt} \quad (2.98)$$

is the time-centroid of $\gamma_{h,z}(t)$. The equivalence of the two right-hand sides of Eq. (2.97) is easily proved by expanding $(t - \bar{t})^2$ in the second line and using Eq. (2.98). Thus, Eq. (2.97) represents the second central moment of the normalized "mass density" $\gamma_{h,z}(t)$ which is analogous to the standard deviation of the "density function" $\gamma_{h,z}(t)$.

It is shown on pp. 100-101 of Ref. 19 [Eq. (A.18) in particular] that the quantity $[-\phi_{\sigma_f^2}^{(0)}(0)/\phi_{\sigma_f^2}^{(2)}(0)]^{1/2}$ is about one-third of the nominal correlation interval of the process $\{\sigma_f^2(t) - \overline{\sigma_f^2}\}$. Consequently, we have

$$\left\{ - \frac{\phi_{\sigma_f^2}^{(2)}(0)}{\phi_{\sigma_f^2}^{(0)}(0)} \left[\frac{\Gamma_{h,z}^{(2)}}{\Gamma_{h,z}^{(0)}} - \left(\frac{\Gamma_{h,z}^{(1)}}{\Gamma_{h,z}^{(0)}} \right)^2 \right] \right\}^{\frac{1}{2}} \approx \frac{\text{"nominal duration" of } \gamma_{h,z}(t)}{\text{"correlation interval" of } \{\sigma_f^2(t) - \overline{\sigma_f^2}\}}, \quad (2.99)$$

where, here, the sign \approx should be read as "is of the order of," and where we have considered the "nominal duration" of $\gamma_{h,z}(t)$ to be about three times the duration of $\gamma_{h,z}(t)$ as measured by the square-root of its second central moment. Consequently, when the correlation interval of the random process

$\{\sigma_f^2(t) - \overline{\sigma_f^2}\}$ is large in comparison with the nominal duration of $\gamma_{h,z}(t)$, the second term in the right-hand note of Eq. (2.96) is negligible and the approximation to $\mu_{\sigma_y^2}^{(2)}$ given by Eq. (2.72) is adequate. Moreover, we see from Eqs. (2.72), (2.96), and (2.99), that the approximation Eq. (2.72) will tend to overestimate the positive quantity $\mu_{\sigma_y^2}^{(2)}$.

Finally, to obtain a physical understanding of the duration of $\gamma_{h,z}(t)$ we consider the case where the power spectrum of the process $\{z(t)\}$ is white — i.e., $\Phi_z(f) = \Phi_z(\cdot) =$ constant. In this case, we see from Eq. (2.65) that

$$\gamma_{h,z}(t) = \Phi_z(\cdot) h^2(t); \quad (2.100)$$

hence, when $\Phi_z(f)$ is a constant over the "passband" of $h(t)$, the above statements pertaining to the duration of $\gamma_{h,z}(t)$ can be interpreted as pertaining to the duration of $h^2(t)$.

In cases where the mean-square velocity of the process $\{\sigma_f^2(t)\}$ does not exist, $|\phi_{\sigma_f^2}^{(2)}(0)|$ does not exist as may be seen from Eq. (2.93), and the series expansion Eq. (2.85) and the two-term approximation Eq. (2.96) are of little use. An alternative series expansion of $\mu_{\sigma_y^2}^{(2)}$ is derived in Appendix D.

This alternative expansion has the advantage that the first

correction term to the limiting case of $\mu_{\sigma_y^2}^{(2)}$ given by Eq.

(2.72) does not require the existence of the mean-square velocity of the process $\{\sigma_f^2(t)\}$. The alternative expansion derived in Appendix D also is more amenable to calculations based on numerical estimation of the power spectrum and autocorrelation function of $\{\sigma_f^2(t)\}$ than the expansion, Eq. (2.85).

In this section, we have provided a detailed methodology for estimating the aircraft response probability density function from appropriate characterizations of the turbulence excitation. Comparable techniques are developed in Sec. 6 for threshold mean exceedance rates of the aircraft response. For cases where fluctuations in $\sigma_f(t)$ in our turbulence model of Eqs. (1.2) and (1.3) are negligible over durations comparable with the duration of the aircraft impulse response $h(t)$, expressions for mean exceedance rates provided on pp. 36 to 46 of Ref. 19 are applicable.

MAXIMUM LIKELIHOOD ESTIMATION OF THE INTEGRAL SCALE AND INTENSITY OF von KARMAN TURBULENCE

Here, we shall derive an optimum method for estimation of the integral scale and intensity of a well-behaved turbulence record with no appreciable low-frequency component $w_s(t)$ present. The vertical time history shown in Fig. 17 illustrates such a record. To develop the method, we shall assume that the functional form of the power spectrum of the record is known — e.g., the transverse or longitudinal von Karman spectral forms. The problem, then, is to develop an optimal method for estimating the parameters that determine the spectrum. In the case of the above two von Karman forms, two parameters — the integral scale L and intensity σ — completely determine the spectrum. Several *ad hoc* methods for estimating values of the integral scale are described on pp. 356 to 360 of Ref. 22.

The method derived below is based on the intuitively appealing procedure called the method of maximum likelihood — e.g., Ref. 38 — which was originally introduced by Gauss. This method is usually treated in the context of optimal estimation of the parameters — e.g., mean and variance — of a probability density of known functional form from a random sample "drawn" from the density. As we show below, it also may be used to estimate the parameters L and σ of a turbulence record whose power spectrum is of known functional form.

Maximum Likelihood Estimation of the Parameter in a Probability Density Function

Let us first briefly review the method in its usual context. Let $p(x|\theta)$ be the probability density of a random variable x , where θ is a parameter in the density function, such as the mean value. Let X_1, X_2, \dots, X_n be n samples drawn from the population whose probability density is governed by $p(x|\theta)$. The likelihood function is defined as

$$L(X_1, X_2, \dots, X_n | \theta) = \prod_{j=1}^n p(X_j | \theta) \quad , \quad (3.1)$$

which is a function of the n sample values X_1, X_2, \dots, X_n and θ , where the functional form $p(x|\theta)$, considered as a function of x and θ , is assumed known. The problem is to determine the "best estimate" of the unknown parameter θ from the n sample values X_1, X_2, \dots, X_n . When θ is assumed known, the right-hand side of Eq. (3.1) is the joint probability density of the sample X_1, X_2, \dots, X_n . The method of maximum likelihood assumes that when θ is unknown, the "most likely" value of θ is the value that maximizes the probability of the observed sample. That is, when X_1, X_2, \dots, X_n are substituted into the right-hand side of Eq. (3.1), the "most likely" value of θ is the value that yields a maximum value of $L(X_1, X_2, \dots, X_n|\theta)$. Generally, it is most convenient to maximize $\log L(X_1, X_2, \dots, X_n|\theta)$ rather than L itself. Thus, the "most likely" value of θ is obtained by solving the equation

$$\frac{\partial L(X_1, X_2, \dots, X_n|\theta)}{\partial \theta} = 0 \quad . \quad (3.2)$$

When more than one solution to Eq. (3.2) exists, the solution chosen is the one that maximizes L when substituted back into $L(X_1, X_2, \dots, X_n|\theta)$. Equation (3.2) is referred to as a likelihood equation. For Eq. (3.2) to yield a *maximum*, it is necessary that $(\partial^2 L / \partial \theta^2) < 0$ at the value of θ determined by Eq. (3.2).

Joint Probability Density of Unsmoothed Turbulence Spectra

Let us turn now to the problem of estimating the integral scale and intensity from a recorded turbulence time history such as the vertical record shown in Fig. 17 which exhibits no discernible slow component $w_s(t)$. To begin with, we shall assume that the record is ^sa time history of finite duration drawn from a stationary Gaussian process; later, we shall argue that the final result is not particularly sensitive to the stationary assumption. Furthermore, we shall assume that the duration T of the record is sufficiently large so that no appreciable bias distortion of its spectrum is caused by operating only with a finite segment of T secs duration. That is, we shall assume that for appropriate (unknown) choices of integral scale and intensity, the *expected value* of the power spectrum of our finite segment of duration T is equal to the actual power spectrum of a record of infinite duration.

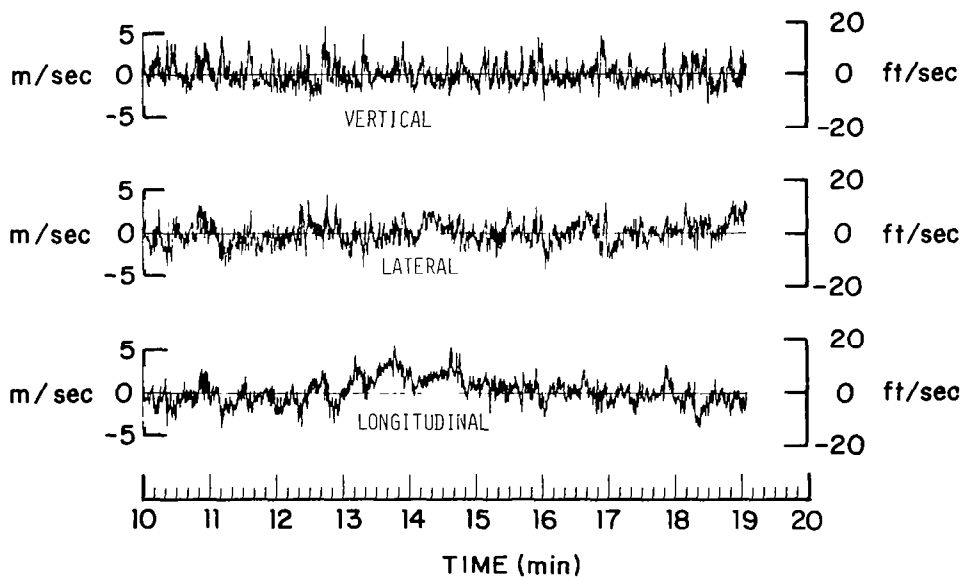
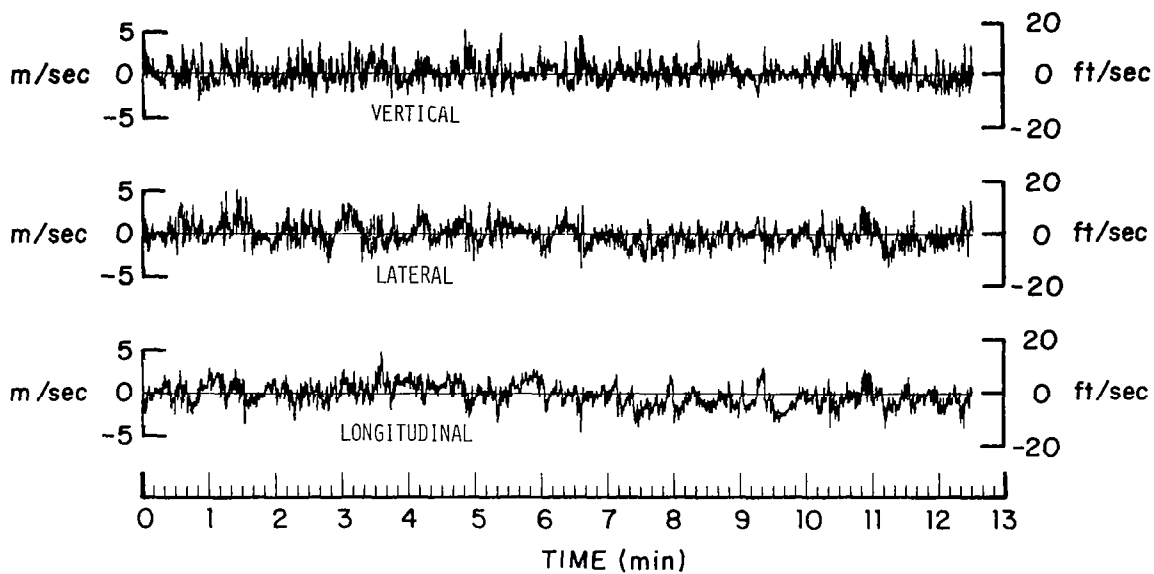


FIG. 17. LOW-ALTITUDE TURBULENCE RECORDS UNDER CONVECTIVE CONDITIONS. [AIRCRAFT SPEED 129 m/sec (422 ft/sec).] (Ref. 23, Fig. 4, p. 282.)

The problem, then, is to estimate L and σ^2 from a segment T secs long drawn from a stationary Gaussian process. The power spectral density of the process is of known functional form, and is determined by the two parameters L and σ^2 . Our interest, of course, is to determine the values of L and σ^2 for the underlying process from which our finite sample has been drawn. Thus, we must consider the statistical properties of a conceptual ensemble of stationary Gaussian segments T secs long that are drawn from the underlying process.

From *each* such segment, we first generate, conceptually, a periodic function by repeating our T sec segment end-on-end. That is, if we denote a typical Gaussian segment by $w(t)$, $-\frac{T}{2} \leq t < \frac{T}{2}$, then from each such segment we generate a periodic function satisfying

$$w(t+pT) = w(t) \quad , \quad p=0, \pm 1, \pm 2, \dots \quad (3.3)$$

Since each such function in our new ensemble is periodic with period T , we may consider the statistics of its complex Fourier series coefficients c_n , $n=0, \pm 1, \pm 2, \dots$ which are the complex amplitudes of the Fourier series components occurring at frequencies of $f=\pm n/T$, $n=0, 1, 2, \dots$ — i.e.,

$$w(t) = \sum_{n=-\infty}^{\infty} c_n e^{i2\pi nt/T} \quad , \quad (3.4)$$

where

$$c_n = \frac{1}{T} \int_{-T/2}^{T/2} w(t) e^{-i2\pi nt/T} dt \quad (3.5a)$$

$$= a_n - i b_n \quad , \quad (3.5b)$$

where a_n and b_n are real. First, we note that since the original process $w(t)$ is Gaussian, the complex Fourier coefficients c_n must be jointly Gaussian complex variables since the operation on $w(t)$ described by Eq. (3.5a) is a linear transformation. In particular, for a given value of n , the probability density of the real and imaginary parts

a_n and b_n of the complex coefficients c_n must be governed by a joint normal probability density on the phase plane. Furthermore, since the original process $w(t)$ is stationary, there can be no preference between the cosine and sine components a_n and b_n ; that is, for every harmonic n , the stationary property of $w(t)$ requires that the joint probability density of the cosine and sine coefficients on the phase plane be rotationally invariant. It follows that a_n and b_n must be uncorrelated and that each is governed by a normal probability density with the same mean value of zero and variance of, say, $\sigma_n^2/2$. For a discussion of the bivariate normal density, see, for example, pp. 147 and 148 of Davenport and Root [39].

Consider, now, the Fourier transform of a typical turbulence segment — which we now defined to be zero outside the interval $(-T/2) \leq t < (T/2)$. From the sampling theorem in the frequency domain — e.g., p. 33 of Woodward [40] — we see that at the values of $f_n = n/T$ described above, the Fourier transform of our turbulence segment is T times the values of the complex Fourier series coefficients c_n . This fact also is immediately evident from Eq. (3.5a). Furthermore, we see from the same sampling theorem that the Fourier transform of our truncated segment is completely determined by the complex coefficients c_n , $n=0, \pm 1, \pm 2, \dots$.

Consider, now, the statistical properties of an *estimate* of the power spectral density of the turbulence process, where the estimate is the so-called periodogram defined as

$$S(f) \triangleq \frac{1}{T} \left| \int_{-T/2}^{T/2} w(t) e^{-i2\pi f t} dt \right|^2 \quad (3.6)$$

— e.g., p. 107 of Davenport and Root [39]. At frequencies $f_n = n/T$, $S(f)$ is related to the Fourier series coefficients of Eq. (3.5) by

$$\begin{aligned} S(f_n) &\equiv S_n = \frac{1}{T} |Tc_n|^2 = T|c_n|^2 \\ &= T[a_n^2 + b_n^2] , \quad n=1, 2, \dots \end{aligned} \quad (3.7)$$

According to the above comments, a_n and b_n are independent, and each is governed by a normal probability density with zero mean and variance $\rho_n^2/2$. Hence, the random variable

$$I_n \triangleq a_n^2 + b_n^2 \quad (3.8)$$

is governed by the exponential probability density function

$$p(I_n) = \begin{cases} \frac{1}{\rho_n^2} e^{-I_n/\rho_n^2} & , I_n > 0 \\ 0 & , I_n < 0 \end{cases} \quad (3.9)$$

as is shown, for example, on p. 53 of Lawson and Uhlenbeck [41], who, incidentally, incorrectly refer to the density of Eq. (3.9) as a Rayleigh density. It follows from Eqs. (3.7) to (3.9) that values of S_n also are governed by an exponential density function, say,

$$p(S_n) = \begin{cases} \lambda_n e^{-\lambda_n S_n} & , S_n \geq 0 \\ 0 & , S_n < 0 \end{cases} \quad (3.10)$$

where $\lambda_n = 1/(T\rho_n^2)$. Finally, we note that the random variables S_n , $n = 0, 1, 2, \dots$ are mutually independent since the real and imaginary parts of the complex Fourier series coefficients c_n , from which the S_n are generated, are uncorrelated Gaussian variates. A proof of the fact that all pairs of random variables a_n, a_m , $n \neq m$; b_n, b_m , $n \neq m$; and a_n, b_m , all n, m , are uncorrelated for $n, m \geq 0$ is provided in Appendix E. Also see p. 119 of Goldman [42].

Our original time segment $w(t)$, $(-T/2) \leq t < (T/2)$ from which $S(f)$ is generated by Eq. (3.6) can be reconstructed

from its complex Fourier series coefficients $c_n = a_n - ib_n$, $n = 0, 1, 2, \dots$. However, we have shown above that for all $m, n \geq 0$, all nonidentical pairs of a_n and b_m are statistically independent. Furthermore, the random variables a_n and b_n both are governed by a normal probability density with zero mean and the same variance $(\sigma_n^2/2) = (1/2\lambda_n T)$, which is uniquely determined by λ_n , $n = 0, 1, 2, \dots$ and T . Consequently, all statistical information about a segment of $w(t)$, $(-T/2) \leq t \leq (T/2)$, is contained in the sequence of coefficients λ_n , $n = 0, 1, 2, \dots$. It follows that the sequence of probability densities $p(S_n)$, $n = 0, 1, 2, \dots$, given by Eq. (3.10) must contain all possible statistical information about the periodogram $S(f)$ defined by Eq. (3.6) since it provides a complete statistical description of the random process $w(t)$, $(-T/2) \leq t \leq (T/2)$, from which $S(f)$ is generated.

Since all S_n are independent for $n > 0$, the joint probability density function of the "vector random variable" $\{S_1, S_2, \dots, S_N\}$ is

$$p(S_1, S_2, \dots, S_N) = \begin{cases} \prod_{j=1}^N \lambda_j e^{-\lambda_j S_j}, & S_j \geq 0, j=1, 2, \dots, N \\ 0 & S_j < 0, j=1, 2, \dots, N \end{cases} \quad (3.11)$$

where we have used Eq. (3.10), and where N can, in principle, be infinity. Finally, we note that λ_j is the reciprocal of the expected value of S_j — as is easily shown in Eq. (3.10):

$$\lambda_j = \frac{1}{E\{S_j\}}, \quad j=1, 2, \dots, N. \quad (3.12)$$

Equation (3.11) is the joint probability density function of samples S_j of the periodogram defined by Eq. (3.6) at frequency values of

$$f_j = j/T, \quad j=1, 2, \dots, N, \quad (3.13)$$

where the periodogram samples taken at the frequencies defined by Eq. (3.13) contain all of the information in the periodogram. Equation (3.13) relates the λ_j in Eq. (3.11) to the expected values of the unsmoothed power spectral density samples $S_j = S(f_j)$.

Likelihood Equations for a General Class of Turbulence Spectra

We may rewrite Eq. (3.11) as

$$p(S_1, S_2, \dots, S_N) = \lambda_1 \lambda_2 \dots \lambda_N e^{-(\lambda_1 S_1 + \lambda_2 S_2 + \dots + \lambda_N S_N)}, \quad (3.14)$$

where, for convenience, we have left out the statement that the right-hand side is valid only for $S_j > 0$, $j = 1, 2, \dots, N$. The logarithm to the base e of Eq. (3.14) is

$$\begin{aligned} \ln[p(S_1, S_2, \dots, S_N)] &= \\ &= [\ln(\lambda_1 \lambda_2 \dots \lambda_N)] - [\lambda_1 S_1 + \lambda_2 S_2 + \dots + \lambda_N S_N] \\ &= - \left\{ [\ln(\bar{S}_1 \bar{S}_2 \dots \bar{S}_N)] + \left[\frac{S_1}{\bar{S}_1} + \frac{S_2}{\bar{S}_2} + \dots + \frac{S_N}{\bar{S}_N} \right] \right\}, \end{aligned} \quad (3.15)$$

where, in going to the last line, we have used Eq. (3.12) and the notation

$$\bar{S}_j \equiv E\{S_j\}, \quad j=1, 2, \dots, N. \quad (3.16)$$

Let us now introduce a class of power spectrum functional forms

$$E\{S_j\} \equiv \bar{S}_j = \sigma^2 L F_j(L), \quad j=1, 2, \dots, N, \quad (3.17)$$

where σ is the rms value of the turbulence, L is the integral scale, and for a given frequency index j , $F_j(L)$ is a function of L but is independent of σ . Equation (3.17) includes the (two-sided) von Karman transverse and longitudinal spectra - e.g., pp. 83 and 93 of Ref. 18 -

$$\Phi_{KT}(k) = \sigma^2 L \frac{1+188.75 L^2 k^2}{[1+70.78 L^2 k^2]^{11/6}} \quad (3.18)$$

$$\Phi_{KL}(k) = \sigma^2 L \frac{2}{[1+70.78 L^2 k^2]^{5/6}} \quad , \quad (3.19)$$

where we have used wavenumber k instead of frequency f as is conventional in turbulence work. According to Eqs. (3.17) to (3.19), for the *von Karman transverse spectrum*, we have

$$F_j(L) = \frac{1+188.75 L^2 k_j^2}{[1+70.78 L^2 k_j^2]^{11/6}} \quad , \quad (3.20)$$

and for the *von Karman longitudinal spectrum*, we have

$$F_j(L) = \frac{2}{[1+70.78 L^2 k_j^2]^{5/6}} \quad . \quad (3.21)$$

In fact, it is easy to show that all spectral forms depending on a single integral scale parameter L must take the form of Eq. (3.17).

When Eq. (3.17) is substituted into Eq. (3.15), we have

$$\begin{aligned} \ln[p(S_1, S_2, \dots, S_N)] = \\ - \left\{ \ln[(\sigma^2 L)^N F_1(L) F_2(L) \dots F_N(L)] \right. \\ \left. + \frac{1}{\sigma^2 L} \left[\frac{S_1}{F_1(L)} + \frac{S_2}{F_2(L)} + \dots + \frac{S_N}{F_N(L)} \right] \right\} \quad , \end{aligned}$$

or

$$\begin{aligned}
\ln[p(S_1, S_2, \dots, S_N)] = & \\
& - \left\{ N \ln(\sigma^2 L) + \ln F_1(L) + \ln F_2(L) + \dots + \ln F_N(L) \right. \\
& \left. + \frac{1}{\sigma^2 L} \left[\frac{S_1}{F_1(L)} + \frac{S_2}{F_2(L)} + \dots + \frac{S_N}{F_N(L)} \right] \right\} . \quad (3.22)
\end{aligned}$$

Equation (3.22) is to be maximized with respect to σ and L . However, from the functional form of the right-hand side of Eq. (3.22) we see that it is more convenient to treat the right-hand side of Eq. (3.22) as a function of the two parameters $(\sigma^2 L)$ and L instead of σ and L . For a given observation (i.e., sample) of our periodogram (S_1, S_2, \dots, S_N) , the values of $(\sigma^2 L)$ and L that maximize the right-hand side of Eq. (3.22) also are the values of σ and L that maximize it — see, for example, p. 257 of Freeman [43]. Let us therefore differentiate Eq. (3.22) with respect to $(\sigma^2 L)$ and L and set the resulting expressions equal to zero:

$$\begin{aligned}
& \frac{\partial \ln[p(S_1, S_2, \dots, S_N; \sigma^2 L, L)]}{\partial (\sigma^2 L)} = \\
& - \left\{ \frac{N}{\sigma^2 L} - \frac{1}{(\sigma^2 L)^2} \left[\frac{S_1}{F_1(L)} + \frac{S_2}{F_2(L)} + \dots + \frac{S_N}{F_N(L)} \right] \right\} \\
& = \frac{-N}{(\sigma^2 L)^2} \left[\sigma^2 L - \frac{1}{N} \sum_{j=1}^N \frac{S_j}{F_j(L)} \right] \\
& = \frac{1}{(\sigma^2 L)^2} \sum_{j=1}^N \left[\frac{S_j}{F_j(L)} - \sigma^2 L \right] , \quad (3.23)
\end{aligned}$$

and

$$\begin{aligned}
& \frac{\partial \ln[p(S_1, S_2, \dots, S_N; \sigma^2 L, L)]}{\partial L} = \\
& - \left\{ \frac{1}{F_1(L)} \frac{dF_1(L)}{dL} + \frac{1}{F_2(L)} \frac{dF_2(L)}{dL} + \dots + \frac{1}{F_N(L)} \frac{dF_N(L)}{dL} \right. \\
& \left. - \frac{1}{\sigma^2 L} \left[\frac{S_1}{F_1^2(L)} \frac{dF_1(L)}{dL} + \frac{S_2}{F_2^2(L)} \frac{dF_2(L)}{dL} + \dots + \frac{S_N}{F_N^2(L)} \frac{dF_N(L)}{dL} \right] \right\} \\
& = \left[- \sum_{j=1}^N \frac{1}{F_j(L)} \frac{dF_j(L)}{dL} \right] + \left[\sum_{j=1}^N \frac{S_j}{\sigma^2 L F_j^2(L)} \frac{dF_j(L)}{dL} \right] \\
& = \frac{1}{\sigma^2 L} \sum_{j=1}^N \left[\frac{d}{dL} \ln F_j(L) \right] \left[\frac{S_j}{F_j(L)} - \sigma^2 L \right], \quad (3.24)
\end{aligned}$$

where parametric dependence of $p(S_1, S_2, \dots, S_N)$ on $\sigma^2 L$ and L has been indicated in the arguments in the left-hand sides of Eqs. (3.23) and (3.24) for clarity. Setting Eq. (3.23) equal to zero yields our likelihood equation for $\sigma^2 L$:

$$\sigma^2 L = \frac{1}{N} \sum_{j=1}^N \frac{S_j}{F_j(L)}. \quad (3.25)$$

Setting Eq. (3.24) equal to zero yields our likelihood equation for L after substitution of Eq. (3.25):

$$\sum_{j=1}^N \left[\frac{d}{dL} \ln F_j(L) \right] \left[\frac{S_j}{F_j(L)} - \frac{1}{N} \sum_{i=1}^N \frac{S_i}{F_i(L)} \right] = 0. \quad (3.26)$$

Equations (3.25) and (3.26) are a pair of likelihood equations for L and $\sigma^2 L$ that involve the periodogram values S_j , $j = 1, 2, \dots, N$ as parameters. Equation (3.26) involves only one unknown, L , and therefore may be solved first. Once L is obtained from Eq. (3.26), Eq. (3.25) may be solved for σ^2 . If we consider our turbulence sample $w(t)$ to be a function of time of duration T , then the values of S_j are those obtained from Eq. (3.6) at values of $f_j = j/T$.

$$S_j = \frac{1}{T} \left| \int_{-T/2}^{T/2} w(t) e^{-i2\pi j t/T} dt \right|^2, \quad (3.27)$$

and the values of $F_j(L)$ are, in the cases of the von Karman transverse and longitudinal spectra, the values given by Eqs. (3.20) and (3.21) after substitution of

$$k_j = j/T. \quad (3.28)$$

In interpreting the resulting values of L as integral scales, care must be taken to properly account for the speed of the aircraft. A method for solution of Eq. (3.26) is described in Appendix F.

Discussion. We can gain some insight into the general likelihood equations (3.25) and (3.26) by examining conditions that their solutions must satisfy. If we divide Eq. (3.25) by $\sigma^2 L$ and substitute Eq. (3.17) into the resulting expression, we obtain after minor rearrangement:

$$\frac{1}{N} \sum_{j=1}^N \left[\frac{S_j}{E\{S_j\}} - 1 \right] = 0. \quad (3.29)$$

Furthermore, if we divide Eq. (3.26) by $N\sigma^2 L$ and combine the resulting expression with Eq. (3.25) and define

$$G_j(L) \triangleq \frac{d}{dL} \ln F_j(L), \quad (3.30)$$

we obtain

$$\frac{1}{N} \sum_{j=1}^N G_j(L) \left[\frac{S_j}{E\{S_j\}} - 1 \right] = 0 \quad . \quad (3.31)$$

Equations (3.29) and (3.31) both are of the same general form — i.e., solutions L and $\sigma^2 L$ to the likelihood equations (3.25) and (3.26) set weighted averages of $[S_j/E\{S_j\}] - 1$ equal to zero. Since the standard deviation of the exponential probability density of Eq. (3.10) is equal to its mean value $E\{S_j\} = 1/\lambda_j$, we see that the standard deviation of each quantity $[S_j/E\{S_j\}]$ in Eqs. (3.29) and (3.31) is equal to the same value of unity.

Finally, we note that Eqs. (3.25) and (3.26) can be written in integral form

$$\sigma^2 L = \frac{1}{k_2 - k_1} \int_{k_1}^{k_2} \frac{S(k)}{F(k;L)} dk \quad (3.32)$$

$$\int_{k_1}^{k_2} \left[\frac{d}{dL} \ln F(k;L) \right] \left[\frac{S(k)}{F(k;L)} - \frac{1}{k_2 - k_1} \int_{k_1}^{k_2} \frac{S(k')}{F(k';L)} dk' \right] dk = 0 \quad , \quad (3.33)$$

where continuous wavenumber k has taken the place of the discrete index j , and where $S(k) = S_j$ and $F(k;L) = F_j(L)$.

Likelihood Equations for von Karman Transverse and Longitudinal Spectra

To specialize Eqs. (3.25) and (3.26) to the von Karman transverse and longitudinal spectral forms, we require the functions $F_j(L)$ defined by Eq. (3.17) and the derivatives of their logarithms with respect to L , Eq. (3.30):

von Karman transverse

$$F_j(L) \equiv F(k_j;L) = \frac{1+188.75 L^2 k_j^2}{[1+70.78 L^2 k_j^2]^{11/6}} \quad (3.34)$$

$$G_j(L) \equiv \frac{d}{dL} \ln F(k_j; L) = \frac{1}{L} \frac{117.97 L^2 k_j^2 (1 - 188.75 L^2 k_j^2)}{(1 + 70.78 L^2 k_j^2)(1 + 188.75 L^2 k_j^2)} \quad (3.35)$$

von Karman longitudinal

$$F_j(L) \equiv F(k_j; L) = \frac{2}{[1 + 70.78 L^2 k_j^2]^{5/6}} \quad (3.36)$$

$$G_j(L) \equiv \frac{d}{dL} \ln F(k_j; L) = - \frac{1}{L} \frac{117.97 L^2 k_j^2}{1 + 70.78 L^2 k_j^2} \quad (3.37)$$

The spectra, Eq. (3.17), associated with the $F_j(L)$ above are two-sided spectra satisfying

$$\sigma^2 = \int_{-\infty}^{\infty} \sigma^2 L F(k; L) dk \quad , \quad (3.38)$$

where k is wavenumber in cycles/unit distance, and L is the integral scale of the longitudinal spectrum which is twice the integral scale of the transverse spectrum — e.g., p. 425 of Houbolt [16]. The exact values of the constants in Eqs. (3.34) to (3.37) are the left-hand sides of

$$25\pi \left[\frac{\Gamma(4/3)}{\Gamma(11/6)} \right]^2 = 70.78 \quad (3.39)$$

$$\frac{200}{3}\pi \left[\frac{\Gamma(4/3)}{\Gamma(11/6)} \right]^2 = 188.75 \quad (3.40)$$

$$\frac{125}{3}\pi \left[\frac{\Gamma(4/3)}{\Gamma(11/6)} \right]^2 = 117.97 \quad (3.41)$$

— see p. 83 of Ref. 18.

Discussion. The von Karman transverse and longitudinal functions $F(k;L)$ of Eqs. (3.34) and (3.36) are shown in Figs. 18 and 19, and the weighting functions $G_j(L)$ (multiplied by L) of Eqs. (3.35) and (3.37) are shown in Figs. 20 and 21. We note first from Figs. 18 and 19 that the functions $F(k_j;L) \equiv F_j(L)$ appearing in Eq. (3.26) are independent of L in the neighborhood near $k = 0$. Hence, spectrum samples S_j near $k = 0$ contain no useful information for obtaining L from the likelihood equation (3.26) — this fact is reflected in the weight function $G_j(L)$ in that $G_j(L)$ is zero near $k = 0$. We further see that

$$G_j(L) = \frac{d}{dL} \ln F_j(L) = \frac{1}{F_j(L)} \frac{dF_j(L)}{dL} \quad (3.42)$$

changes sign at $kL = (188.75)^{-\frac{1}{2}}$ for the transverse case shown in Fig. 20. However, from Eq. (3.42), we see that $G_j(L)$ is zero at values of L where dF_j/dL is zero; that is, at points where $F_j(L)$ is independent of L . Here again, we see that $G_j(L)$ provides zero weighting to the values of k in Eq. (3.26) where $F_j(L)$ is independent of L . Finally, we note from Figs. 20 and 21 that for large values of k , values of $G_j(L)$ approach a constant value. Correspondingly, in Figs. 18 and 19, we see for these same large values of k that variations in $\log F_j(L)$ with L are independent of k . Finally, there is nothing in the behavior of the likelihood equations or the functions shown in Figs. 18 to 21 to indicate that solutions obtained for L and $\sigma^2 L$ should be particularly sensitive to the hypothesis that values of S_j are mutually independent; hence, we would expect the likelihood equations (3.25) and (3.26) to also yield good results for nonstationary records with slowly varying nonstationary behavior [18].

The statistical confidence of estimates of L and $\sigma^2 L$ will be discussed in a later section of this report.

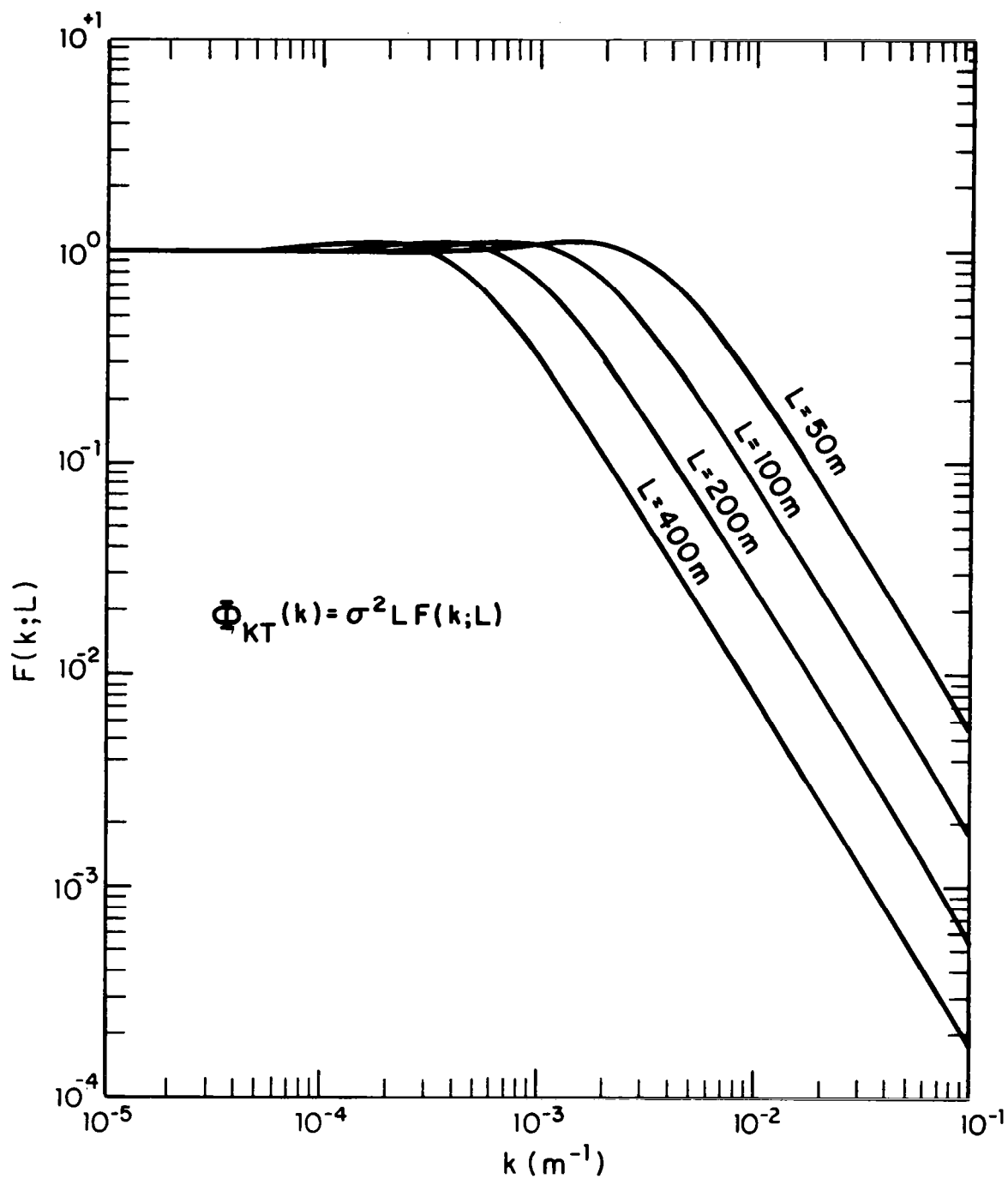


FIG. 18. FUNCTION $F(k;L)$ GIVEN BY EQ. (3.34) FOR von KARMAN TRANSVERSE SPECTRUM $\Phi_{KT}(k)$.

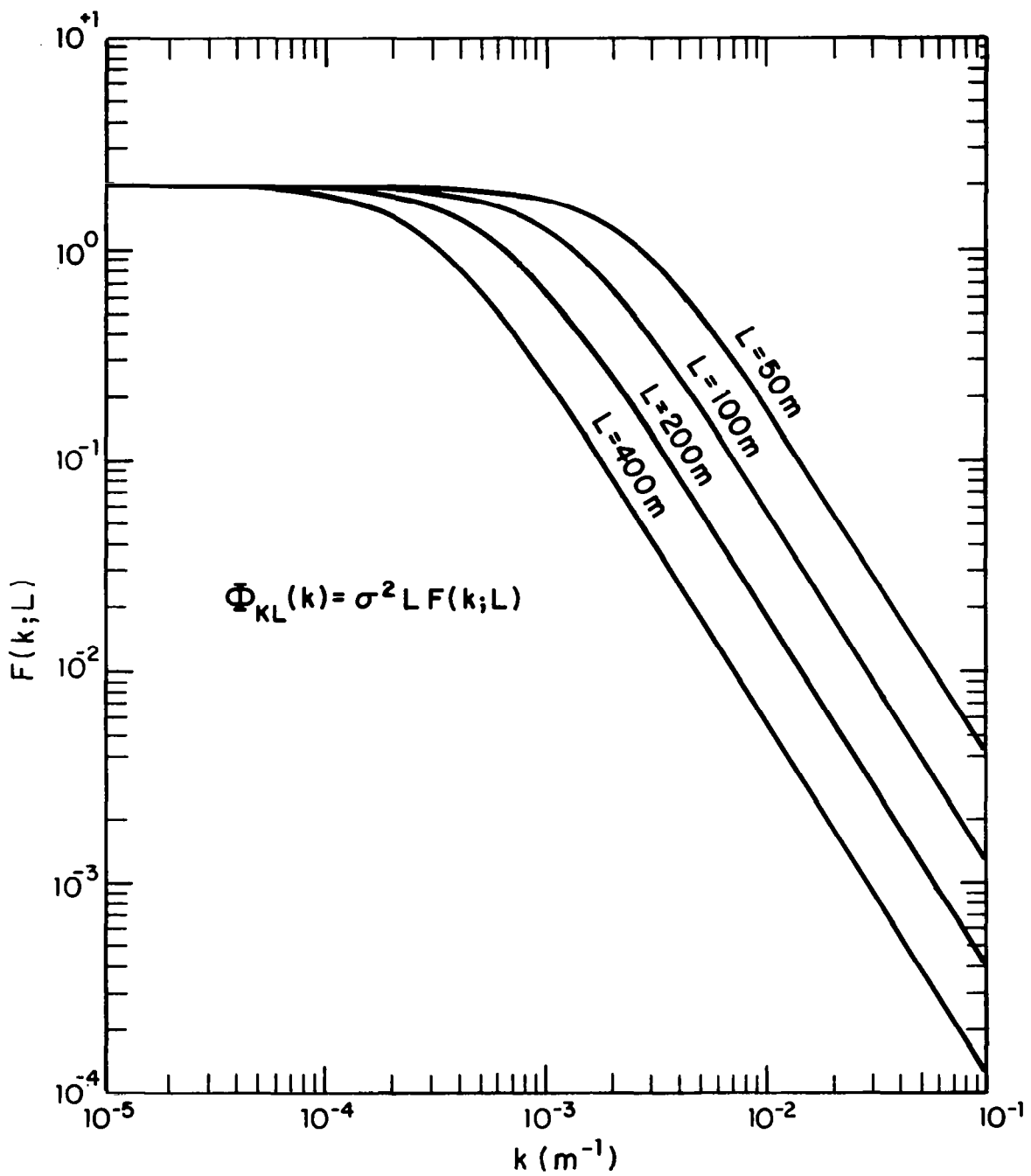


FIG. 19. FUNCTION $F(k;L)$ GIVEN BY EQ. (3.36) FOR von KARMAN LONGITUDINAL SPECTRUM $\Phi_{KL}(k)$.

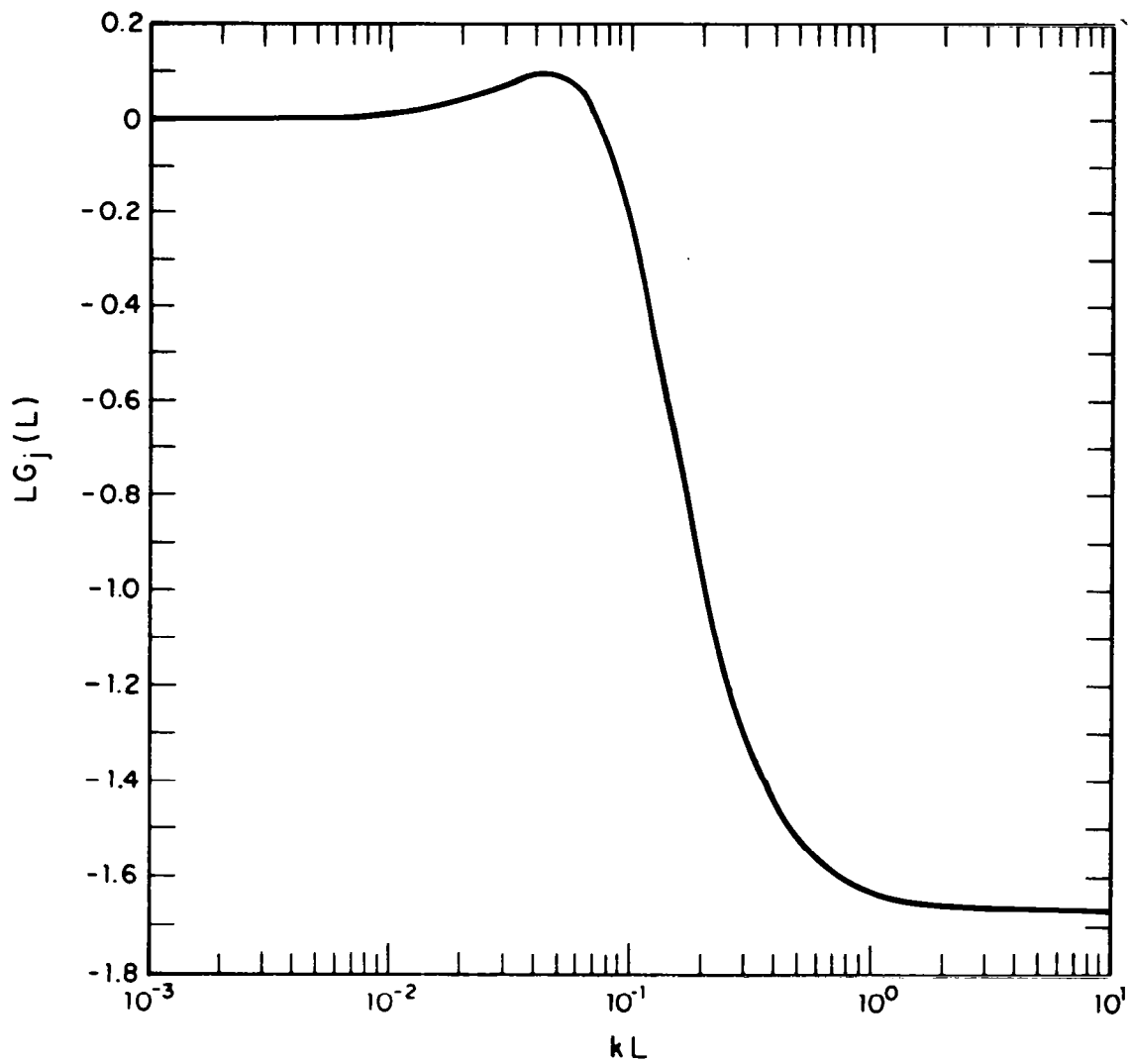


FIG. 20. FUNCTION $LG_j(L)$ GIVEN BY EQ. (3.35) FOR von KARMAN TRANSVERSE SPECTRUM.

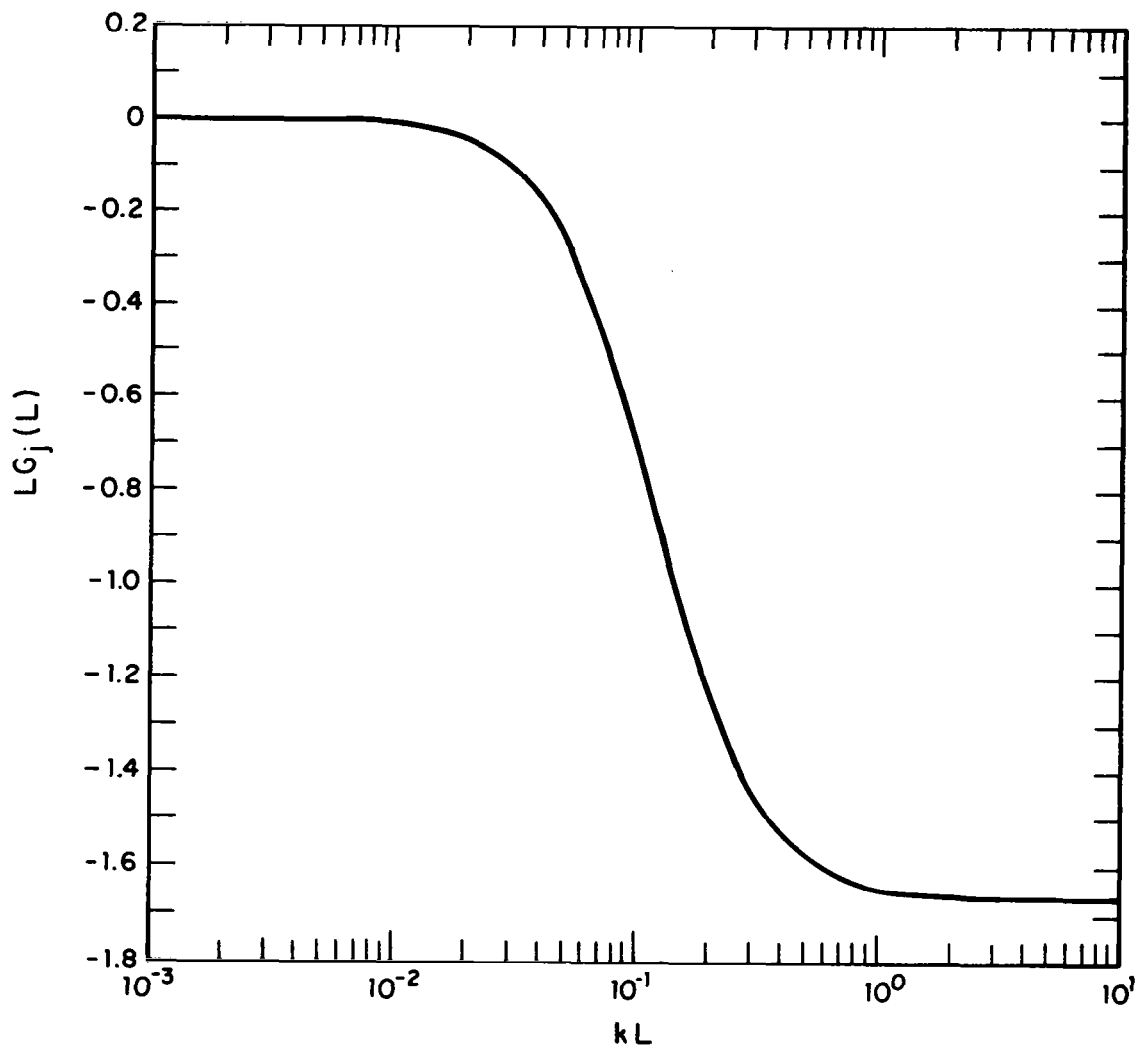


FIG. 21. FUNCTION $LG_j(L)$ GIVEN BY EQ. (3.37) FOR von KARMAN LONGITUDINAL SPECTRUM.

CONSTRAINED LEAST-SQUARES ESTIMATION OF TURBULENCE AUTOCORRELATION FUNCTION PARAMETERS

General Approach

In the method for estimation of the integral scale and intensity described in Sec. 3.4, it is assumed that the turbulence time histories are drawn from a stationary, Gaussian random process possessing either the von Karman transverse or longitudinal power spectral forms. As we have discussed in Sec. 1 of this report, many turbulence histories recorded in practice have an additive low frequency (long wavenumber) component superimposed on what may be described as ordinary von Karman turbulence — e.g., see Figs. 4 through 6. Consequently, the likelihood equations derived in Sec. 3 should not be used with such records.

When a nonnegligible fraction of the mean-square velocity of a record is contained in the low frequency component $w_s(t)$, we have decided to use, jointly, the autocorrelation function and the power spectral density of the record to estimate the integral scale and intensity of the von Karman component. After examination of the autocorrelation functions computed from a number of velocity histories recorded in the MAT Project [30], it became evident that over the range of the delay variable ξ , say $0 \leq \xi \leq \xi_H$, where the autocorrelation function of the von Karman component $w_f(t)$ is nonnegligible, the autocorrelation function of the low frequency component $w_s(t)$ could almost always be represented with reasonable accuracy by a low-order polynomial — e.g., see Fig. 5 of Ref. 19. A reliable model of the *power spectral density* of the low frequency component $w_s(t)$ of comparable generality and simplicity is not immediately evident. Therefore, let us express our autocorrelation model as

$$\phi(\xi) \triangleq \overline{\sigma_f^2} \phi_K(\xi; L) + \sum_{i=0}^m a_i \xi^i, \quad 0 \leq \xi \leq \xi_H, \quad (4.1)$$

where $\overline{\sigma_f^2} \equiv E\{\sigma_f^2\}$ is the mean-square value of the von Karman "fast" component $w_f(t)$, $\phi_K(\xi; L)$ is the appropriate (transverse or longitudinal) von Karman autocorrelation function

normalized so that $\phi_K(0;L) \equiv 1$, L is the integral scale of the von Karman component, and

$$\phi_{w_s}(\xi) = \sum_{i=0}^m a_i \xi^i, \quad 0 \leq \xi \leq \xi_H, \quad (4.2)$$

is the polynomial approximation to the autocorrelation function of the slow component $w_s(t)$ which is valid over $0 \leq \xi \leq \xi_H$, and which can contain odd as well as even powers of ξ . Equations (4.1) and (4.2) are consistent with Eqs. (1.18) (1.29) discussed earlier.

We shall evaluate $\overline{\sigma_f^2}$ and L by minimizing the integral squared-difference E between the autocorrelation function model, Eq. (4.1), and the empirically obtained autocorrelation function $R(\xi)$ that Eq. (4.1) represents:

$$E \triangleq \int_0^{\xi_H} \left\{ R(\xi) - \overline{\sigma_f^2} \phi_K(\xi;L) - \sum_{i=0}^m a_i \xi^i \right\}^2 d\xi, \quad (4.3)$$

where the minimization procedure will be *constrained* as follows. By definition, the "slow" turbulence component $w_s(t)$ contains predominantly low frequencies in comparison with the von Karman "fast" component $w_f(t)$. This comment suggests that there usually will exist a wavenumber k_ℓ such that for values of $k > k_\ell$, the wavenumber spectrum of a turbulence record $w(t)$ will be, for practical purposes, dominated by contributions from the von Karman component $w_f(t)$ only. In this wavenumber region, we therefore may use the likelihood equation (3.25) which we now write as

$$\overline{\sigma_f^2} L - \frac{1}{N} \sum_{j=1}^N \frac{S_j}{F_j(L)} = 0 \quad (4.4)$$

to constrain the minimization of E . In using this constraint, we shall consider Eq. (4.4) to determine L as a function of $\overline{\sigma_f^2}$. Furthermore, only the spectrum values S_j for which $k_j > k_\ell$ will be used in Eq.(4.4); that is, the summation over j in Eq. (4.4) will include only the wavenumbers $k_j > k_\ell$.

We may explicitly include the constraint, Eq. (4.4), in our formulation by rewriting Eq. (4.3) as

$$E = \int_0^{\xi_H} \left\{ R(\xi) - \sigma^2 \phi_K[\xi; L(\sigma^2)] - \sum_{i=0}^m a_i \xi^i \right\}^2 d\xi, \quad (4.5)$$

where, for simplicity, we have used the abbreviated notation

$$\sigma^2 \equiv \overline{\sigma_f^2}, \quad (4.6)$$

and where, now, L is considered as the function of σ^2 determined by Eq. (4.4) from wavenumber spectrum values S_j for $k_j \geq k_\ell$. Trade-offs between choices of ξ_H and m are discussed in Appendix G. A method that extends the model of Eq. (4.1) to the entire interval $0 \leq \xi < \infty$, thereby permitting computation of the integral scale and power spectrum of the slow component $w_s(t)$, is discussed in Appendix H.

Discussion. By this juncture, the reader has undoubtedly asked why the second likelihood equation (3.26) is not being used for the range of values $k_j \geq k_\ell$ to determine L . Equations (3.25) and (3.26) both assume that the spectrum values S_j are obtained from turbulence sample functions possessing von Karman spectra. Thus, in choosing k_ℓ , we must be reasonably sure that for all $k_j \geq k_\ell$, the contribution from $w_s(t)$ to this portion of the spectrum is negligible. By careful examination of spectra measured in the MAT Project [23], we have concluded that for a large fraction of atmospheric turbulence records, such values of k_ℓ fall at the approximate location of the knee of the von Karman portion of the spectrum. Let us examine the behavior of the likelihood equation (3.26) for cases where the smallest value of j in the summation occurs slightly above this value at a wavenumber $k_j \approx 1/L$. Examination of Figs. 20 and 21 shows that for values of $k \geq 1/L$, $G_j(L)$ is approximately equal to the constant value

$$G_j(L) \sim -\frac{5}{3L}, \quad k_j \geq L^{-1}, \quad (4.7)$$

where the value of $-5/(3L)$ can be verified for both the von Karman transverse and longitudinal spectra from Eqs. (3.35) and (3.37). If we divide Eq. (3.26) by N and substitute the above asymptotic value of $G_j(L)$ into the resulting expression, we obtain

$$- \frac{5}{3L} \left\{ \frac{1}{N} \sum_{j=1}^N \left[\frac{S_j}{F_j(L)} - \frac{1}{N} \sum_{i=1}^N \frac{S_i}{F_i(L)} \right] \right\} = 0$$

or

$$- \frac{5}{3L} \left\{ \left[\frac{1}{N} \sum_{j=1}^N \frac{S_j}{F_j(L)} \right] - \frac{1}{N} \sum_{j=1}^N \left[\frac{1}{N} \sum_{i=1}^N \frac{S_i}{F_i(L)} \right] \right\} = 0, \quad (4.8)$$

which is satisfied identically for any value of L . Hence, when the smallest usable value k_ℓ of k_j is L^{-1} or larger, the likelihood equation (3.26) is satisfied identically for all values of L and therefore is useless. For somewhat smaller values of k_ℓ , Eq. (3.26) will still yield unreliable results. Hence, in essence, we have replaced Eq. (3.26) by minimization of the parameter E described by Eq. (4.5).

On the other hand, the first likelihood equation (3.25) is perfectly well behaved when the only usable values of k_j are the values $k_j > L^{-1}$. In fact, for the records that we shall discuss in the applications portion of this work, the likelihood equation (3.25) yields results with excellent statistical reliability. It therefore has been retained as the constraint $L = L(\sigma^2)$ in the minimization of Eq. (4.5).

There exist two additional reasons for using Eq. (4.4) as a constraint in minimization of the integral squared-difference E given by Eq. (4.5). First, we note that when the degree m of the polynomial of Eq. (4.2) is taken too large, the von Karman autocorrelation function $\overline{\sigma_f^2} \phi_K(\xi; L)$ can be represented quite well by the polynomial of Eq. (4.2). When this is the case, minimization of E — given by Eq. (4.3) — leads to set of ill-conditioned equations for L , $\overline{\sigma_f^2}$, and a_0, a_1, \dots, a_m . On the other hand, when Eq. (4.4) is included as a constraint in the minimization of E — as indicated by Eq. (4.5) — the von Karman component $\sigma^2 \phi_K[\xi; L(\sigma^2)]$ remains dependent on only one parameter σ^2 in the minimization; consequently, the range of shapes that $\sigma^2 \phi_K[\xi; L(\sigma^2)]$ can take on in the minimization is greatly reduced, and for a given value of m , the minimization becomes much better conditioned. This is a very important consideration when working with empirical data such as the

empirical autocorrelation function $R(\xi)$ in Eq. (4.5). We shall see in the applications portion of this work how this improvement in conditioning yields improved results in certain specific cases.

The second reason for using the constraint $L = L(\sigma^2)$ of Eq. (4.4) is that we shall want to estimate the wavenumber spectrum of the "slow" component $w_s(t)$ by subtracting our best estimate of the spectrum of the von Karman component from the overall spectrum of $w(t)$. Since the equation of constraint (4.4) was obtained directly from the wavenumber (or frequency) domain and represents an optimum estimate obtained from the von Karman portion of the spectrum, we would expect to obtain better results for the spectrum of $w_s(t)$ when Eq. (4.4) is included as a constraint in the minimization of E .

Finally, we should comment on why it is *possible* for the minimization of E — as described by Eq. (4.5) and the constraint equation (4.4) — to yield potentially better estimates of σ^2 and L than use of the pair of likelihood equations (3.25) and (3.26) over the range of values of $k_j > k_\ell$ where the wavenumber spectrum is dominated by the spectrum of the von Karman component $w_f(t)$. In setting up the quantity E represented by Eq. (4.5) to be minimized, we have included the representation, Eq. (4.2), of the autocorrelation function of the low-frequency turbulence component, $w_s(t)$. *Equivalently, we have included a representation of some of the information about the "slow" turbulence component $w_s(t)$ that would be found in the wavenumber domain in the region $k < k_\ell$.* We also have included information about the von Karman component that would be found in the same low-wavenumber region $k < k_\ell$. The reason we have chosen to work with the autocorrelation function representation of Eq. (4.1) rather than with a comparable representation of the spectrum of $w(t) = w_s(t) + w_f(t)$ is that it is a simple matter to make intelligent *a priori* choices of ξ_H and m for our representation of $\phi_{w_s}(\xi)$ given by Eq. (4.2) which should lead to good results when the integral squared-difference given by Eq. (4.5) is minimized. A representation of the wavenumber spectrum of $w_s(t)$ of comparable generality, simplicity and amenableness to analysis is not obvious.

Derivation of Algebraic Equations for Autocorrelation Function Parameters

Let us now derive the equations whose solution yields a minimum value of E . That is, from Eq. (4.5), we wish to derive a set of algebraic equations whose solution for σ^2 and a_0, a_1, \dots, a_m minimizes the value of E given by Eq. (4.5), where Eq. (4.4) with $\overline{\sigma_f^2} \equiv \sigma^2$ serves to define L as a function of σ^2 for use in the resulting set of equations. These equations are obtained by setting the derivative of Eq. (4.5) with respect to a_0, a_1, \dots, a_m and σ^2 equal to zero, where L is considered as a function of σ^2 :

$$\frac{\partial E}{\partial a_j} = 0, \quad j=0,1,\dots,m \quad \frac{\partial E}{\partial \sigma^2} = 0. \quad (4.9a,b)$$

Differentiating Eq. (4.5) with respect to a_j , we obtain

$$\frac{\partial E}{\partial a_j} = -2 \int_0^{\xi_H} \left\{ R(\xi) - \sigma^2 \phi_K[\xi; L(\sigma^2)] - \sum_{i=0}^m a_i \xi^i \right\} \xi^j d\xi, \quad j=0,1,\dots,m. \quad (4.10)$$

Furthermore, differentiating Eq. (4.5) with respect to σ^2 while treating L as a function of σ^2 , we obtain

$$\begin{aligned} \frac{\partial E}{\partial \sigma^2} = & -2 \int_0^{\xi_H} \left\{ R(\xi) - \sigma^2 \phi_K[\xi; L(\sigma^2)] - \sum_{i=0}^m a_i \xi^i \right\} \\ & \times \left\{ \sigma^2 \frac{\partial \phi_K}{\partial L} \frac{dL}{d\sigma^2} + \phi_K[\xi; L(\sigma^2)] \right\} d\xi. \end{aligned} \quad (4.11)$$

Setting Eqs. (4.10) and (4.11) equal to zero yields the set of $m + 2$ *nonlinear* algebraic equations for a_0, a_1, \dots, a_m and σ^2 which can be written as

$$\sigma^2 \int_0^{\xi_H} \xi^j \phi_K[\xi; L(\sigma^2)] d\xi + \sum_{i=0}^m a_i \int_0^{\xi_H} \xi^{i+j} d\xi = \int_0^{\xi_H} \xi^j R(\xi) d\xi, \quad j=0,1,\dots,m \quad (4.12)$$

and

$$\begin{aligned} & \sigma^2 \int_0^{\xi_H} \left\{ \sigma^2 \frac{\partial \phi_K[\xi; L(\sigma^2)]}{\partial L} \frac{dL}{d\sigma^2} + \phi_K[\xi; L(\sigma^2)] \right\} \\ & \quad \times \phi_K[\xi; L(\sigma^2)] d\xi \\ & + \sum_{i=0}^m a_i \int_0^{\xi_H} \left\{ \sigma^2 \frac{\partial \phi_K[\xi; L(\sigma^2)]}{\partial L} \frac{dL}{d\sigma^2} + \phi_K[\xi; L(\sigma^2)] \right\} \xi^i d\xi \\ & = \int_0^{\xi_H} \left\{ \sigma^2 \frac{\partial \phi_K[\xi; L(\sigma^2)]}{\partial L} \frac{dL}{d\sigma^2} + \phi_K[\xi; L(\sigma^2)] \right\} R(\xi) d\xi, \quad (4.13) \end{aligned}$$

where, from Eqs. (4.4) and (4.6), $L(\sigma^2)$ is defined by the equation

$$\sigma^2 = \frac{1}{N} \sum_{j=1}^N \frac{S_j}{LF_j(L)}, \quad (4.14)$$

where, as indicated earlier, only values of S_j dominated by the von Karman component are included within the summation in Eq. (4.14).

For the von Karman transverse and longitudinal spectra of Eqs. (3.18) and (3.19), let us define the normalized spectrum $\bar{\Phi}(\bar{k})$, where

$$\bar{k} \triangleq Lk, \quad (4.15)$$

by

$$\Phi_K(k) = \sigma^2 L \bar{\Phi}_K(Lk) \quad ; \quad (4.16)$$

where the overbars denote normalized spectra and normalized wavenumber. Then, for either transverse or longitudinal spectra, the corresponding transverse or longitudinal autocorrelation function is

$$\begin{aligned} \sigma^2 \phi_K(\xi) &= \int_{-\infty}^{\infty} \Phi_K(k) e^{i2\pi k \xi} dk \\ &= \sigma^2 L \int_{-\infty}^{\infty} \bar{\Phi}_K(Lk) e^{i2\pi Lk \xi/L} dk \\ &= \sigma^2 \int_{-\infty}^{\infty} \Phi_K(\bar{k}) e^{i2\pi \bar{k} \xi/L} d\bar{k} \end{aligned} \quad (4.17)$$

If we define a normalized length measure by

$$\bar{\xi} \triangleq \xi/L \quad (4.18)$$

and a normalized autocorrelation function by

$$\bar{\phi}_K(\bar{\xi}) \triangleq \int_{-\infty}^{\infty} \bar{\Phi}_K(\bar{k}) e^{i2\pi \bar{k} \bar{\xi}} d\bar{k} \quad , \quad (4.19)$$

then, according to Eqs. (4.17) to (4.19), we may express $\phi_K(\xi)$ as

$$\phi_K[\xi; L(\sigma^2)] \equiv \phi_K(\xi) = \bar{\phi}_K(\xi/L) = \bar{\phi}_K(\bar{\xi}) \quad . \quad (4.20)$$

Consider, now, the derivatives of ϕ_K with respect to L which appear in Eq. (4.13). According to Eq. (4.18), we have

$$\frac{\partial \bar{\xi}}{\partial L} = - \frac{\xi}{L^2} ; \quad (4.21)$$

hence, from Eq. (4.21), it follows that

$$\begin{aligned} \frac{\partial \phi_K[\xi; L(\sigma^2)]}{\partial L} &= \frac{d\bar{\phi}_K(\bar{\xi})}{d\bar{\xi}} \frac{\partial \bar{\xi}}{\partial L} \\ &= - \frac{\xi}{L^2} \frac{d\bar{\phi}_K(\bar{\xi})}{d\bar{\xi}} . \end{aligned} \quad (4.22)$$

Turning to the integrals that appear in Eq. (4.13), let us define

$$\begin{aligned} I_1(L) &\triangleq - \int_0^{\xi_H} \frac{\partial \phi_K[\xi; L(\sigma^2)]}{\partial L} \phi_K[\xi; L(\sigma^2)] d\xi \\ &= \int_0^{\xi_H/L} \bar{\xi} \bar{\phi}'_K(\bar{\xi}) \bar{\phi}_K(\bar{\xi}) d\bar{\xi} , \end{aligned} \quad (4.23)$$

where Eqs. (4.18), (4.20), and (4.22) were used in going to the second line, and where

$$\bar{\phi}'_K(\bar{\xi}) \equiv \frac{d\bar{\phi}_K(\bar{\xi})}{d\bar{\xi}} . \quad (4.24)$$

The second integral that we require is

$$\begin{aligned}
I_2(L) &\triangleq \frac{1}{L} \int_0^{\xi_H} \phi_K^2[\xi; L(\sigma^2)] d\xi \\
&= \int_0^{\xi_H/L} \bar{\phi}_K^2(\bar{\xi}) d\bar{\xi} \quad , \quad (4.25)
\end{aligned}$$

where Eqs. (4.18) and (4.20) have been used in going to the second line. Next, we require

$$\begin{aligned}
I_3(j, L) &\triangleq - \frac{1}{L^j} \int_0^{\xi_H} \xi^j \frac{\partial \phi_K[\xi; L(\sigma^2)]}{\partial L} d\xi \\
&= \int_0^{\xi_H/L} \bar{\xi}^{1+j} \bar{\phi}'_K(\bar{\xi}) d\bar{\xi} \quad , \quad j=0,1,\dots,m \quad (4.26)
\end{aligned}$$

where Eqs. (4.18) and (4.22) have been used. From Eq. (4.13), we also require

$$\begin{aligned}
I_4(j, L) &\triangleq \frac{1}{L^{1+j}} \int_0^{\xi_H} \xi^j \phi_K[\xi; L(\sigma^2)] d\xi \\
&= \int_0^{\xi_H/L} \bar{\xi}^j \bar{\phi}_K(\bar{\xi}) d\bar{\xi} \quad , \quad j=0,1,\dots,m \quad , \quad (4.27)
\end{aligned}$$

where Eqs. (4.18) and (4.20) have been used. We next require

$$\begin{aligned}
I_5(L) &\triangleq - \frac{1}{R(0)} \int_0^{\xi_H} \frac{\partial \phi_K[\xi; L(\sigma^2)]}{\partial L} R(\xi) d\xi \\
&= \frac{1}{R(0)L^2} \int_0^{\xi_H} \xi \bar{\phi}'_K(\xi/L) R(\xi) d\xi \quad (4.28a)
\end{aligned}$$

$$= \frac{1}{R(0)} \int_0^{\xi_H/L} \bar{\xi} \bar{\phi}'_K(\bar{\xi}) R(L\bar{\xi}) d\bar{\xi} , \quad (4.28b)$$

where Eqs. (4.18) and (4.22) have been used. We further require

$$\begin{aligned}
I_6(L) &\triangleq \frac{1}{R(0)L} \int_0^{\xi_H} \phi_K[\xi; L(\sigma^2)] R(\xi) d\xi \\
&= \frac{1}{R(0)L} \int_0^{\xi_H} \bar{\phi}_K(\xi/L) R(\xi) d\xi \quad (4.29a)
\end{aligned}$$

$$= \frac{1}{R(0)} \int_0^{\xi_H/L} \bar{\phi}_K(\bar{\xi}) R(L\bar{\xi}) d\bar{\xi} , \quad (4.29b)$$

where Eqs. (4.18) and (4.20) have been used. Two additional integrals are required for use in Eq. (4.12):

$$\begin{aligned}
I_7(j, L) &\triangleq \frac{1}{L^{1+j}} \int_0^{\xi_H} \xi^j d\xi \\
&= \frac{(\xi_H/L)^{1+j}}{1+j} , \quad j=0, 1, \dots, 2m \quad (4.30)
\end{aligned}$$

and

$$I_8(j, L) \triangleq \frac{1}{R(0)L^{1+j}} \int_0^{\xi_H} \xi^j R(\xi) d\xi \quad (4.31a)$$

$$= \frac{1}{R(0)} \int_0^{\xi_H/L} \bar{\xi}^j R(L\bar{\xi}) d\bar{\xi}, \quad j=0,1,\dots,m \quad (4.31b)$$

where Eq. (4.18) has been used. The above quantities I_1 through I_8 are dimensionless.

Using the definitions of Eqs. (4.23) to (4.31), we can rewrite Eqs. (4.13) and (4.12) as

$$\begin{aligned} & \sigma^2 \left[-\sigma^2 \frac{dL}{d\sigma^2} I_1(L) + L I_2(L) \right] \\ & + \sum_{i=0}^m a_i \left[-\sigma^2 \frac{dL}{d\sigma^2} L^i I_3(i, L) + L^{1+i} I_4(i, L) \right] \\ & = -R(0) \sigma^2 \frac{dL}{d\sigma^2} I_5(L) + R(0)L I_6(L) \end{aligned} \quad (4.32)$$

and

$$\begin{aligned} \sigma^2 L I_4(j, L) + \sum_{i=0}^m a_i L^{1+i} I_7(i+j, L) &= R(0)L I_8(j, L), \\ j=0,1,\dots,m. \end{aligned} \quad (4.33)$$

Equations (4.32) and (4.33) for $j=0,1,\dots,m$ constitute a set of $m+2$ nonlinear simultaneous algebraic equations for σ^2 and a_0, a_1, \dots, a_m . For a given value of σ^2 , Eq. (4.14) (inverted) determines the quantity $L(\sigma^2)$. Equation (4.14) also determines $dL/d\sigma^2$:

$$\frac{dL}{d\sigma^2} = \left(\frac{d\sigma^2}{dL} \right)^{-1} = \left\{ -\frac{1}{N} \sum_{j=1}^N \frac{[F_j(L) + L F_j'(L)] S_j}{L^2 F_j^2(L)} \right\}^{-1}, \quad (4.34)$$

which yields $dL/d\sigma^2$ for a given value of $L(\sigma^2)$.

Matrix Form of Algebraic Equations for Autocorrelation Function Parameters

For numerical solution, the set of $m + 2$ equations (4.32) and (4.33) can be written in the matrix form

$$\begin{aligned} A_{11}(\sigma^2)y_1 + A_{12}(\sigma^2)y_2 + \dots + A_{1,m+2}(\sigma^2)y_{m+2} &= x_1(\sigma^2) \\ A_{21}(\sigma^2)y_1 + A_{22}(\sigma^2)y_2 + \dots + A_{2,m+2}(\sigma^2)y_{m+2} &= x_2(\sigma^2) \\ &\vdots \\ A_{m+2,1}(\sigma^2)y_1 + A_{m+2,2}(\sigma^2)y_2 + \dots + A_{m+2,m+2}(\sigma^2)y_{m+2} &= x_{m+2}(\sigma^2) \end{aligned} \quad (4.35a)$$

or, more concisely, as

$$\sum_{\ell=1}^{m+2} A_{k\ell}(\sigma^2)y_\ell = x_k(\sigma^2) \quad , \quad k=1,2,\dots,m+2 \quad (4.35b)$$

where

$$y_1 \equiv \sigma^2, \quad y_\ell \equiv a_{\ell-2} \quad \text{for } \ell=2,3,\dots,m+2 \quad (4.36)$$

$$\begin{aligned} x_1 &\equiv R(0) \left[-\sigma^2 \frac{dL}{d\sigma^2} I_5(L) + L I_6(L) \right] \\ x_k &\equiv R(0)L I_8(k-2,L) \quad \text{for } k=2,3,\dots,m+2 \end{aligned} \quad (4.37)$$

$$A_{11} \equiv -\sigma^2 \frac{dL}{d\sigma^2} I_1(L) + L I_2(L)$$

$$A_{1\ell} \equiv -\sigma^2 \frac{dL}{d\sigma^2} L^{\ell-2} I_3(\ell-2, L) + L^{\ell-1} I_4(\ell-2, L) ,$$

$$\ell=2, 3, \dots, m+2$$

$$A_{k1} \equiv L I_4(k-2, L) , \quad k=2, 3, \dots, m+2$$

$$A_{k\ell} \equiv L^{\ell-1} I_7(k+\ell-4, L) , \quad k=2, 3, \dots, m+2; \quad \ell=2, 3, \dots, m+2.$$

(4.38)

Method of Solution

Scaling of matrix equations. The $m + 2$ equations (4.35) were scaled before their solution was obtained — using the method suggested on pp. 118–119 of Hamming [44]. This scaling is carried out as follows:

1. Consider a new array of "coefficients" which is the array A_{ij} in Eq. (4.35) with the right-hand side added as an additional column

$$\begin{array}{cccc} A_{11} & A_{12} & A_{1,m+2} & x_1 \\ A_{21} & A_{22} & \dots & A_{2,m+2} & x_2 \\ \vdots & \vdots & & \vdots & \vdots \\ A_{m+2,1} & A_{m+2,2} & & A_{m+2,m+2} & x_{m+2}. \end{array}$$

2. Call the above new array A'_{ij} where i ranges from 1 to $m + 2$ and j ranges from 1 to $m + 3$.

3. Form a new array \tilde{A}'_{ij} from the array A'_{ij} where for every $i=1, 2, \dots, m+2$ and $j=1, 2, \dots, m+3$

$$\tilde{A}'_{ij} = 2^{r_i + c_j + M} A'_{ij} .$$

(4.39)

Formulas for r_i , c_j , and M will be given in item 6 below.

4. Let $\tilde{y}_1, \tilde{y}_2, \dots, \tilde{y}_{m+2}$ be the solution to a new set of matrix equations which may be written in terms of our original notation as

$$\begin{bmatrix} 2^{r_i+c_j+M} & \\ & A_{ij} \end{bmatrix} \{\tilde{y}_j\} = \begin{bmatrix} 2^{r_i+c_{m+3}+M} & \\ & x_i \end{bmatrix} . \quad (4.40)$$

5. Multiplication of every element by 2^M does nothing to the solution. Multiplication of every row by 2^{r_i} does nothing to the solution. The effects of multiplication of the right-hand side by $2^{c_{m+3}}$ can be eliminated by dividing every element by $2^{c_{m+3}}$. Therefore, the relationship between y_j and \tilde{y}_j is

$$\frac{2^{c_j}}{2^{c_{m+3}}} \tilde{y}_j = y_j . \quad (4.41a)$$

In other words, for every j the solution y_j of the original equations (4.35) is obtained from the solution \tilde{y}_j of Eqs. (4.40) by

$$y_j = 2^{c_j - c_{m+3}} \tilde{y}_j , \quad j=1,2,\dots,m+2 . \quad (4.41b)$$

6. To determine values for r_i , c_j , and M we first form for every pair of values i,j where $i=1,2,\dots,m+2$ and $j=1,2,\dots,m+3$ the quantity

$$d_{ij} = \frac{\log_{10} |A_{ij}|}{\log_{10} 2} \quad (4.42)$$

where $|A_{ij}|$ denotes the absolute value of A_{ij} , and where $A_{i,m+3} \equiv x_i$ as in item 1. Values of M , r_i , and c_j are then computed by

$$M = \frac{-1}{(m+2)(m+3)} \sum_{j=1}^{m+3} \sum_{i=1}^{m+2} d_{ij} \quad (4.43)$$

$$r_i = \frac{-1}{m+3} \sum_{j=1}^{m+3} (d_{ij} + M) \quad (4.44)$$

$$c_j = \frac{-1}{m+2} \sum_{i=1}^{m+2} (d_{ij} + M) \quad (4.45)$$

Initialization of computation. Steps in preparing for solution of the matrix equations (4.40) are:

1. Compute the unsmoothed power spectral density of the record of interest. Call the (unsmoothed) computed spectrum values S_i . Pick lower and upper wavenumbers k_l and k_u which define a wavenumber region $k_l < k < k_u$ where we are confident that the spectrum is dominated by the von Karman component of the turbulence. For values of S_i that fall within this wavenumber region, determine for a set of equally spaced values of L_j the quantity

$$\sigma_j^2 \equiv \sigma^2(L_j) = \frac{1}{N} \sum_{i=1}^N \frac{S_i}{L_j F_i(L_j)} \quad (4.46)$$

as defined by Eq. (4.14), where $F_i(L)$ is given by Eq. (3.20) for transverse (vertical or lateral) records and by Eq. (3.21) for longitudinal records. When $\sigma^2(L_j)$ is computed for a range of equally spaced values of L_j , we have σ^2 as a numerical function of the integral scale L_j . This tabulation is to be considered as determining L as a function of σ^2 . Values of L falling between tabulated values of L_j are obtained by interpolation.

2. Values of $dL/d\sigma^2$ also are required as a function of σ^2 . In our computations, we used the approximation

$$\frac{dL}{d\sigma^2} \approx \frac{L_{j+1} - L_j}{\sigma_{j+1}^2 - \sigma_j^2} \quad (4.47)$$

where the values of L and σ^2 in the right-hand side of Eq. (4.47) are those computed by Eq. (4.46). A more accurate procedure would be to compute $dL/d\sigma^2$ by Eq. (4.34) for the same values of L_j used in Eq. (4.46).

3. From the autocorrelation function $R(\xi)$ of the record under consideration, and for the same values of L_j used in Eqs. (4.46) and (4.47), the integrals of I_1 through I_8 given by Eqs. (4.23) through (4.31) are computed numerically. Each of these integrals is a function of the integral scale L . Thus, I_1 through I_8 is to be computed for each of the above mentioned equi-spaced value of L_j . The values of the integrals are then tabulated. Computation of these integrals requires the functions $\bar{\phi}_K(\bar{\xi})$ and $\bar{\phi}'_K(\bar{\xi})$ which, for von Karman transverse and longitudinal spectra, are:

for von Karman transverse spectra (vertical and lateral components):

$$\bar{\phi}_K(\bar{\xi}) = \frac{2^{2/3}}{\Gamma(1/3)} (\beta \bar{\xi})^{1/3} \left[K_{1/3}(\beta \bar{\xi}) - \frac{\beta}{2} \bar{\xi} K_{-2/3}(\beta \bar{\xi}) \right] \quad (4.48)$$

$$\bar{\phi}'_K(\bar{\xi}) = \frac{\beta}{2^{1/3} \Gamma(1/3)} \left[(\beta \bar{\xi})^{4/3} K_{1/3}(\beta \bar{\xi}) - \frac{8}{3} (\beta \bar{\xi})^{1/3} K_{-2/3}(\beta \bar{\xi}) \right] \quad (4.49)$$

for von Karman longitudinal spectra:

$$\bar{\phi}_K(\bar{\xi}) = \frac{2^{2/3}}{\Gamma(1/3)} (\beta \bar{\xi})^{1/3} K_{1/3}(\beta \bar{\xi}) \quad (4.50)$$

$$\bar{\phi}'_K(\bar{\xi}) = \frac{-2^{2/3} \beta}{\Gamma(1/3)} (\beta \bar{\xi})^{1/3} K_{-2/3}(\beta \bar{\xi}) , \quad (4.51)$$

where

$$\beta \triangleq \frac{2\sqrt{\pi} \Gamma(11/6)}{5 \Gamma(4/3)} , \quad (4.52)$$

$K_{1/3}$ and $K_{-2/3}$ are modified Bessel functions of the second kind of order $1/3$ and $-2/3$, and $\Gamma(\cdot)$ is the gamma function. Values of the Bessel functions were read off of the tabulation on p. 228 of Ref. 45, where we note that $K_{-2/3}(x) = K_{2/3}(x)$. Values of $\bar{\xi} \bar{\phi}'_K(\bar{\xi})$ were tabulated rather than values of $\bar{\phi}'_K(\bar{\xi})$, because $\bar{\phi}_K(\bar{\xi})$ has a singularity at $\bar{\xi} = 0$ whereas $\bar{\xi} \bar{\phi}'_K(\bar{\xi})$ tends to zero as $\bar{\xi} \rightarrow 0$. Values of I_1 through I_8 required for $L \neq L_j$ were obtained by interpolation between the tabulated values of the I 's.

Solution for σ^2 and a_0, a_1, \dots, a_m . Once $L(\sigma^2)$, $dL/d\sigma^2$, and I_1 through I_8 are tabulated for a predetermined set of equi-spaced values of $L = L_j$, the solution to the set of equations (4.40) can be obtained. This solution yields the solution to the set (4.35) by Eq. (4.41b). Although the scaled equations (4.40) were actually the set solved, we shall describe the solution procedure in terms of Eqs. (4.35).

The coefficients A_{kl} and the right-hand sides x_k of Eqs. (4.35) are functions of the unknown variance $\sigma^2 \equiv y_1$ of the von Karman component. Thus, Eqs. (4.35) were solved by trial and error. From a plot of the autocorrelation function of the record $R(\xi)$, a rough estimate of $\sigma^2 \equiv \overline{\sigma_f^2}$ is easily obtained. This value of σ^2 determines $L = L(\sigma^2)$ and $dL/d\sigma^2$ which are obtained by interpolation from predetermined tabulated values of these quantities. I_1 through I_8 also are determined by interpolation from predetermined values tabulated as functions of $L = L_j$. Consequently, once the initial value of σ^2 has been chosen, the coefficients A_{ij} , $i, j = 1, 2, \dots, m+2$ and right-hand side x_i , $i = 1, 2, \dots, m+2$ in Eqs. (4.35) are determined. Equations (4.35) are then solved, the solution being a new value of σ^2 and a_0, a_1, \dots, a_m [see Eq. (4.36)]. Using the new value of σ^2 , the coefficients A_{ij} , $i, j = 1, 2, \dots, m+2$ and right-hand side x_i , $i = 1, 2, \dots, m+2$ are again evaluated, and the set of Eqs. (4.35) is solved again. The new solution yields a new value of σ^2 and values of a_0, a_1, \dots, a_m . By comparing the *solution* values of σ^2 with the *input* values of σ^2 for these two solutions, a new trial value of σ^2 is chosen and the set of Eqs. (4.35) is solved again after evaluation of the coefficients and the right-hand side using the new trial value of σ^2 . In carrying out this procedure, we terminated the process when the input and solution values of

σ^2 were in agreement to three significant figures. The final values of $\sigma^2 \equiv \overline{\sigma_1^2}$ and $L(\sigma^2)$ determine the spectrum of the von Karman component of the turbulence.

VARIANCE OF MAXIMUM LIKELIHOOD ESTIMATES OF von KARMAN TURBULENCE PARAMETERS

In Sec. 3, a pair of equations (3.25) and (3.26) was derived whose solution yields maximum likelihood estimates of $\sigma^2 L$ and L from spectrum samples S_j , $j = 1, 2, \dots, N$. These spectrum samples are to be computed from turbulence velocity records whose (expected) power spectral densities are assumed to be of the von Karman transverse or longitudinal forms. Since individual turbulence records are stochastic functions of time, *estimates* of $\sigma^2 L$ and L computed from such records also are stochastic. Thus, in any particular application the solution $\sigma^2 L$, L to the pair of equations (3.25) and (3.26) is a two-dimensional random variable whose joint probability density function depends on the duration of the turbulence sample function from which the estimate of $\sigma^2 L$ and L is obtained. In this section, the probability density functions of maximum likelihood estimates of $\sigma^2 L$ and L are discussed. In addition, explicit formulas are given for the squares of the *coefficients of variation* - i.e., relative variances - of $\sigma^2 L$, L , and σ^2 .

Asymptotic Forms of Probability Density Functions of Turbulence Parameter Estimates

In the "standard" class of problems that employs maximum likelihood estimation; each of the individual probability density functions $\lambda_j e^{-\lambda_j S_j}$, $j = 1, 2, \dots, N$ in Eq. (3.11) is identical. However, this is not true in the present application because the values of the parameters λ_j depend on j - as may be seen from Eqs. (3.12), (3.17), (3.20), and (3.21). Hence, care must be exercised in applying the "standard" results to the present problem.

The "standard" result that is our main interest is the asymptotic (large sample) form of the joint probability density function of our estimates of $\sigma^2 L$ and L obtained by solving Eqs. (3.25) and (3.26). In the standard class of problems mentioned above, such maximum likelihood estimates are jointly normally distributed in the large sample limit - see, e.g., p. 55 of Ref. 38 or p. 155 of Ref. 46. In the present case, we shall appeal to the form of the multidimensional large sample result cited on p. 155 of Ref. 46, which avoids the troublesome problem of bias, and which is the multidimensional extension of the approach used by Cramer on pp. 550 to 504 of Ref. 8.

To justify use of the large sample asymptotic normal form in the present "nonstandard" application, we must show as the total duration T of our turbulence sample function approaches infinity that (1) $\partial \ln p / \partial(\sigma^2 L)$ and $\partial \ln p / \partial L$ are asymptotically jointly normally distributed with zero expected values at the true values of $\sigma^2 L$ and L and (2) sample values of $\partial^2 \ln p / \partial^2(\sigma^2 L)$, $\partial^2 \ln p / \partial(\sigma^2 L) \partial L$, and $\partial^2 \ln p / \partial L^2$ converge to their respective expected values in the sense that *fractional* deviations from their expected values vanish with probability one as $T \rightarrow \infty$ — see pp. 154 and 155 of Ref. 46 and pp. 43 and 55 of Ref. 38.

Consider requirement (1) first. According to Eqs. (3.23) and (3.24), $\partial \ln p / \partial(\sigma^2 L)$ and $\partial \ln p / \partial L$ both are linear combinations of the independent random variables (S_j / F_j) , $j = 1, 2, \dots, N$. From Eqs. (3.10), (3.12), and (3.17), it follows that each random variable (S_j / F_j) is governed by an exponential probability density function with the same (unknown) expected value of $\sigma^2 L$. From this fact, it follows that at the true values of $\sigma^2 L$ and L we have $E\{\partial \ln p / \partial(\sigma^2 L)\} = 0$ and $E\{\partial \ln p / \partial L\} = 0$. To show that the two-dimensional random variable $\{\partial \ln p / \partial(\sigma^2 L), \partial \ln p / \partial L\}$ is asymptotically normally distributed as $T \rightarrow \infty$, we employ the two-dimensional central limit theorem — e.g., pp. 285–287 of Ref. 8. Since, in the case of Eq. (3.24), we are dealing with a weighted sum of independent random variables, where the weighting function is $\frac{d}{dL} \ln F_j(L)$, we first consider the behavior of the sums in Eqs. (3.23) and (3.24) over a typical limited fixed frequency range, say Δf , as $T \rightarrow \infty$. According to Eq. (3.13), if the frequency range Δf is fixed, as $T \rightarrow \infty$ the number of samples within Δf increases indefinitely since the frequency difference between adjacent samples is $1/T$. This behavior is true for every frequency interval Δf over which the histogram $S(f)$ — Eq. (3.6) — is computed. Since the slope of the weight function $\frac{d}{dL} \ln F_j(L)$ is everywhere finite — see Eq. (3.30) and Figs. 20 and 21 — we may choose Δf small enough so that $\frac{d}{dL} \ln F_j(L)$ is essentially constant within Δf . We may now apply the central limit theorem to the contributions of the sums in Eqs. (3.23) and (3.24) within each such frequency interval Δf to show that the contributions from each such Δf are asymptotically jointly normally distributed as $T \rightarrow \infty$. The sums in Eqs. (3.23) and (3.24) over all such disjoint intervals then represent the sums of (nonidentically distributed) independent two-dimensional normal variables which are known to be normally distributed — e.g., pp. 212 and 316 of Ref. 8. Hence, as $T \rightarrow \infty$, the two-dimensional random variable $\{\partial \ln p / \partial(\sigma^2 L), \partial \ln p / \partial L\}$ is asymptotically normally distributed with zero mean value at the true values of $\sigma^2 L$ and L .

Let us now consider requirement (2) above. Forming the partial derivative of Eq. (3.23) with respect to $\sigma^2 L$ gives

$$\frac{\partial^2 \ln[p(S_1, S_2, \dots, S_N; \sigma^2 L, L)]}{\partial (\sigma^2 L)^2} = - \frac{N}{(\sigma^2 L)^2} - \frac{2}{(\sigma^2 L)^3} \sum_{j=1}^N \left[\frac{S_j}{F_j(L)} - \sigma^2 L \right]; \quad (5.1)$$

forming the partial derivative of Eq. (3.23) with respect to L gives

$$\begin{aligned} \frac{\partial^2 \ln[p(S_1, S_2, \dots, S_N; \sigma^2 L, L)]}{\partial (\sigma^2 L) \partial L} &= - \frac{1}{(\sigma^2 L)^2} \sum_{j=1}^N [F_j(L)]^{-2} \frac{dF_j(L)}{dL} S_j \\ &= - \frac{1}{(\sigma^2 L)^2} \sum_{j=1}^N \left[\frac{d}{dL} \ln F_j(L) \right] \frac{S_j}{F_j(L)}; \end{aligned} \quad (5.2)$$

and forming the partial derivative of Eq. (3.24) with respect to L gives

$$\begin{aligned} \frac{\partial^2 \ln[p(S_1, S_2, \dots, S_N; \sigma^2 L, L)]}{\partial L^2} &= \frac{1}{\sigma^2 L} \sum_{j=1}^N \left\{ \left[\frac{d}{dL} \ln F_j(L) \right] \left[- \frac{S_j}{F_j^2(L)} \frac{dF_j(L)}{dL} \right] \right. \\ &\quad \left. + \left[\frac{d^2}{dL^2} \ln F_j(L) \right] \left[\frac{S_j}{F_j(L)} - \sigma^2 L \right] \right\} \end{aligned}$$

or,

$$\begin{aligned}
& \frac{\partial^2 \ln[p(S_1, S_2, \dots, S_N; \sigma^2 L, L)]}{\partial L^2} = \\
& - \frac{1}{\sigma^2 L} \left\{ \sum_{j=1}^N \left[\frac{d}{dL} \ln F_j(L) \right]^2 \frac{S_j}{F_j(L)} \right. \\
& \quad \left. - \sum_{j=1}^N \left[\frac{d^2}{dL^2} \ln F_j(L) \right] \left[\frac{S_j}{F_j(L)} - \sigma^2 L \right] \right\} . \quad (5.3)
\end{aligned}$$

Taking the expected values of each of the above three equations, using

$$E \left\{ \frac{S_j}{F_j(L)} \right\} = \sigma^2 L, \quad (5.4)$$

and introducing an obvious shorthand notation for the left-hand sides gives

$$E \left\{ \frac{\partial^2 \ln p}{\partial (\sigma^2 L)^2} \right\} = - \frac{N}{(\sigma^2 L)^2}, \quad (5.5)$$

$$E \left\{ \frac{\partial^2 \ln p}{\partial (\sigma^2 L) \partial L} \right\} = - \frac{1}{\sigma^2 L} \sum_{j=1}^N \frac{d}{dL} \ln F_j(L), \quad (5.6)$$

and

$$E \left\{ \frac{\partial^2 \ln p}{\partial L^2} \right\} = - \sum_{j=1}^N \left[\frac{d}{dL} \ln F_j(L) \right]^2. \quad (5.7)$$

Let us define the fractional deviation of $\partial^2 \ln p / \partial (\sigma^2 L)^2$ [given by Eq. (5.1)] from its expected value [given by Eq. (5.5)] by ϵ_1 - i.e.,

$$\begin{aligned}\epsilon_1 &\triangleq \frac{[\partial^2 \ln p / \partial (\sigma^2 L)^2] - E\{\partial^2 \ln p / \partial (\sigma^2 L)^2\}}{E\{\partial^2 \ln p / \partial (\sigma^2 L)^2\}} \\ &\equiv \frac{\frac{1}{N} \left([\partial^2 \ln p / \partial (\sigma^2 L)^2] - E\{\partial^2 \ln p / \partial (\sigma^2 L)^2\} \right)}{\frac{1}{N} E\{\partial^2 \ln p / \partial (\sigma^2 L)^2\}},\end{aligned}\quad (5.8)$$

and denote analogous definitions of the fractional deviations of $\partial^2 \ln p / \partial (\sigma^2 L) \partial L$ and $\partial^2 \ln p / \partial L^2$ by ϵ_2 and ϵ_3 respectively. From Eqs. (5.1) to (5.3) and (5.4) to (5.6), it follows directly from the expression for ϵ_1 given by Eq. (5.8) and the corresponding two expressions for ϵ_2 and ϵ_3 that

$$\epsilon_1 = \frac{2}{\sigma^2 L} \frac{1}{N} \sum_{j=1}^N \left[\frac{S_j}{F_j(L)} - \sigma^2 L \right] \quad (5.9)$$

$$\epsilon_2 = \frac{\frac{1}{N} \sum_{j=1}^N \left[\frac{d}{dL} \ln F_j(L) \right] \left[\frac{S_j}{F_j(L)} - \sigma^2 L \right]}{\sigma^2 L \frac{1}{N} \sum_{j=1}^N \frac{d}{dL} \ln F_j(L)} \quad (5.10)$$

and

$$\epsilon_3 = \frac{\frac{1}{N} \sum_{j=1}^N \left\{ \left[\frac{d}{dL} \ln F_j(L) \right]^2 - \left[\frac{d^2}{dL^2} \ln F_j(L) \right] \right\} \left[\frac{S_j}{F_j(L)} - \sigma^2 L \right]}{\sigma^2 L \frac{1}{N} \sum_{j=1}^N \left[\frac{d}{dL} \ln F_j(L) \right]^2}. \quad (5.11)$$

We must show that ϵ_1 , ϵ_2 , and ϵ_3 all approach zero with probability one as our turbulence sample duration $T \rightarrow \infty$. Since the random variables $S_j/F_j(L)$ are each governed by an exponential probability density with mean value $\sigma^2 L$, it follows from the (weak) law of large numbers — e.g., p. 228 of Ref. 47 — that $\epsilon_1 \rightarrow 0$ as $T \rightarrow \infty$ with probability one.

To show that $\varepsilon_2 \rightarrow 0$ as $T \rightarrow \infty$, we note first that, in the limit $T \rightarrow \infty$, the denominator of the right-hand side of Eq. (5.10) approaches the limiting constant value

$$\mathcal{D}_2 \triangleq \frac{\sigma^2 L}{(N_2 - N_1) \Delta k} \int_{N_1 \Delta k}^{N_2 \Delta k} \frac{d}{dL} \ln F(k; L) dk . \quad (5.12)$$

This fact becomes clear when we observe (1) that the denominator in Eq. (5.10) is $\sigma^2 L$ times the average value of the derivative of $\ln F_j(L)$ with respect to L , (2) that, in the large sample limit, dependence of $F_j(L)$ on the discrete variable j is replaced by dependence on the continuous variable k as shown in Eqs. (3.34) and (3.36), and (3) that in taking the limit $T \rightarrow \infty$ in applications, we fix the lower and upper wavenumber limits

$$k_1 = N_1 \Delta k, \quad k_2 = N_2 \Delta k \quad (5.13)$$

where $N = N_2 - N_1$ and increase indefinitely the *density* of sampling points within the interval $k_2 - k_1$ so that the sum approaches the definite integral in Eq. (5.12) of the continuous function

$$G_j(L) = \frac{d}{dL} \ln F(k_j; L) \equiv \frac{d}{dL} \ln F_j(L) \quad (5.14)$$

shown in Figs. 20 and 21 for von Karman transverse and longitudinal spectra.

Consider, now, the numerator of Eq. (5.10). Here, we may appeal directly to the form of the (weak) law of large numbers given on p. 238 of Ref. 47, which applies to sequences of non-identically distributed independent random variables. The numerator in Eq. (5.10) can be expressed as

$$N_2 \triangleq \frac{1}{N} \sum_{j=1}^N x_j , \quad (5.15)$$

where the

$$x_j = \left[\frac{d}{dL} \ln F_j(L) \right] \left[\frac{S_j}{F_j(L)} - \sigma^2 L \right] \quad (5.16)$$

are the random variables of interest. Since $S_j/F_j(L)$ is governed by an exponential probability density with mean $\sigma^2 L$ (independent of j), it follows that the random variable $\{[S_j/F_j(L)] - \sigma^2 L\}$ has zero mean and variance $(\sigma^2 L)^2$. Hence, the random variables defined by Eq. (5.16) have zero mean and variances

$$\sigma_j^2 = (\sigma^2 L)^2 \left[\frac{d}{dL} \ln F_j(L) \right]^2. \quad (5.17)$$

A sufficient condition for N_2 to vanish with probability one as $T \rightarrow \infty$ is - according to Eq. (5.6) on p. 238 of Ref. 47 - that

$$\begin{aligned} \frac{s_N^2}{N^2} &= \frac{1}{N^2} \sum_{j=1}^N \sigma_j^2 \\ &= \frac{(\sigma^2 L)^2}{N^2} \sum_{j=1}^N \left[\frac{d}{dL} \ln F_j(L) \right]^2 \end{aligned} \quad (5.18)$$

vanish as $T \rightarrow \infty$. However, in this limit Eq. (5.18) approaches

$$\begin{aligned} \frac{s_N^2}{N^2} &\rightarrow \frac{(\sigma^2 L)^2}{N(N_2 - N_1) \Delta k} \int_{N_1 \Delta k}^{N_2 \Delta k} \left[\frac{d}{dL} \ln F(k; L) \right]^2 dk \\ &= \frac{(\sigma^2 L)^2}{N(k_2 - k_1)} \int_{k_1}^{k_2} \left[\frac{d}{dL} \ln F(k; L) \right]^2 dk, \end{aligned} \quad (5.19)$$

where we have substituted $N = N_2 - N_1$ and have written the limiting form of the sum on the continuous function $\left[\frac{d}{dL} \ln F(k_j, L) \right]^2$ as an integral as before. The right-hand side of Eq. (5.19) is proportional to $1/N$; hence, (s_N/N) is proportional in the limit to $1/\sqrt{N}$ and therefore to $1/\sqrt{T}$. ϵ_2 therefore approaches zero with probability one as $T \rightarrow \infty$. The argument that ϵ_3 must approach zero with probability one as $T \rightarrow \infty$ is carried out in exactly the same way as that for ϵ_2 .

The maximum likelihood *estimates* of $\sigma^2 L$ and L therefore are asymptotically jointly normally distributed with mean values equal to the true parameter values (denoting estimates of $\sigma^2 L$ and L subscripts 1 and 2 respectively)

$$m_1 = \sigma^2 L, \quad m_2 = L, \quad (5.20 \text{ a,b})$$

and with covariance matrix Λ whose *inverse* is given by

$$\Lambda^{-1} = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}, \quad (5.21)$$

where

$$A_{11} = -E \left\{ \frac{\partial^2 \ln p}{\partial (\sigma^2 L)^2} \right\}, \quad A_{22} = -E \left\{ \frac{\partial^2 \ln p}{\partial L^2} \right\},$$

and

$$A_{12} = A_{21} = -E \left\{ \frac{\partial^2 \ln p}{\partial (\sigma^2 L) \partial L} \right\} \quad (5.22 \text{ a,b,c})$$

(p. 55 of Ref. 38 and p. 155 of Ref. 46). Expressions for the (negatives) of A_{11} , A_{12} , and A_{21} are given by Eqs. (5.5) to (5.7).

It is easy to verify that the inverse of Λ^{-1} is given by

$$\Lambda = \frac{1}{|\Lambda^{-1}|} \begin{bmatrix} A_{22} & -A_{12} \\ -A_{21} & A_{11} \end{bmatrix}, \quad (5.23)$$

where $|\Lambda^{-1}|$ is the determinant of Λ^{-1} ,

$$|\Lambda^{-1}| = A_{11}A_{22} - A_{12}A_{21}, \quad (5.24)$$

and where the form of Eq. (5.23) can be verified by multiplying Eq. (5.21) by (5.23) and using Eq. (5.24):

$$\Lambda^{-1}\Lambda = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} . \quad (5.25)$$

If we write the covariance matrix Λ in the conventional form — e.g., p. 295 of Ref. 8 —

$$\Lambda = \begin{bmatrix} \sigma_1^2 & \rho\sigma_1\sigma_2 \\ \rho\sigma_1\sigma_2 & \sigma_2^2 \end{bmatrix} , \quad (5.26)$$

it follows by comparison of Eqs. (5.22), (5.23), (5.26), and Eqs. (5.5) through (5.7) that we have

$$\sigma_1^2 = \frac{-E\left\{\frac{\partial^2 \ln p}{\partial L^2}\right\}}{|\Lambda^{-1}|} = \frac{\sum_{j=1}^N \left[\frac{d}{dL} \ln F_j(L)\right]^2}{|\Lambda^{-1}|} , \quad (5.27)$$

$$\sigma_2^2 = \frac{-E\left\{\frac{\partial^2 \ln p}{\partial (\sigma^2 L)^2}\right\}}{|\Lambda^{-1}|} = \frac{N}{(\sigma^2 L)^2} , \quad (5.28)$$

and

$$\rho = \frac{E\left\{\frac{\partial^2 \ln p}{\partial (\sigma^2 L) \partial L}\right\}}{\sigma_1 \sigma_2 |\Lambda^{-1}|} = - \frac{\sum_{j=1}^N \frac{d}{dL} \ln F_j(L)}{\sqrt{N \sum_{j=1}^N \left[\frac{d}{dL} \ln F_j(L)\right]^2}} , \quad (5.29)$$

where from Eqs. (5.22), (5.24), and (5.5) through (5.7), we have

$$\begin{aligned}
 |\Lambda^{-1}| &= E \left\{ \frac{\partial^2 \ln p}{\partial (\sigma^2 L)^2} \right\} E \left\{ \frac{\partial^2 \ln p}{\partial L^2} \right\} - \left[E \left\{ \frac{\partial^2 \ln p}{\partial (\sigma^2 L) \partial L} \right\} \right]^2 \\
 &= \frac{1}{(\sigma^2 L)^2} \left\{ N \sum_{j=1}^N \left[\frac{d}{dL} \ln F_j(L) \right]^2 - \left[\sum_{j=1}^N \frac{d}{dL} \ln F_j(L) \right]^2 \right\}. \quad (5.30)
 \end{aligned}$$

In summary, we have shown that solutions to the pair of likelihood equations (3.25) and (3.26) provide estimates to the true values of $\sigma^2 L$ and L that as $T \rightarrow \infty$ are asymptotically governed by a joint normal probability density with mean values equal to the true parameter values $\sigma^2 L$ and L as indicated by Eqs. (5.20 a,b) and with covariance matrix elements given by Eqs. (5.27) to (5.30), where σ_1^2 and σ_2^2 denote, respectively, the variances of our estimates of $\sigma^2 L$ and L , and ρ denotes the correlation coefficient of these estimates. The limiting forms (as $T \rightarrow \infty$) of σ_1^2 , σ_2^2 , and ρ for the von Karman transverse and longitudinal spectral forms will be evaluated in the next section.

Expressions for Covariance Matrix Elements for von Karman Transverse and Longitudinal Spectra

Let us turn now to evaluation of the large T limiting forms of the sums in Eqs. (5.27) to (5.30) which apply to our asymptotic joint normal distribution of the maximum likelihood estimates of $\sigma^2 L$ and L . We shall continue to use the notation established in Sec. 3 — Eq. (3.17) in particular — and at the end of this section we shall specialize the results to the von Karman spectral forms of Eqs. (3.20) and (3.21).

First consider the expression for ρ given by the right-hand side of Eq. (5.29). If we divide both numerator and denominator of Eq. (5.29) by N , we shall require expressions for the two quantities

$$I^{(1)} \triangleq \frac{1}{N} \sum_{j=1}^N \frac{d}{dL} \ln F_j(L) \quad (5.31)$$

and

$$I^{(2)} \triangleq \frac{1}{N} \sum_{j=1}^N \left[\frac{d}{dL} \ln F_j(L) \right]^2. \quad (5.32)$$

Since the right-hand sides of Eqs. (5.27) to (5.30) represent the limiting forms of the variances and correlation coefficient of our estimates of $\sigma^2 L$ and L as $T \rightarrow \infty$, we can, without loss in generality, consider only the limiting forms of the right-hand sides of Eqs. (5.31) and (5.32) as $T \rightarrow \infty$. Denote these limiting forms by

$$I^{(1)} \triangleq \lim_{T \rightarrow \infty} I^{(1)}, \quad I^{(2)} \triangleq \lim_{T \rightarrow \infty} I^{(2)} \quad (5.33 \text{ a,b})$$

and consider $I^{(1)}$ first. Here, we shall consider the record to be of length L where $L = VT$, T being the duration of the record and V being the speed of the aircraft used in measuring the record. Hence, our N wavenumber spectrum samples S_j discussed in Sec. 3 are spaced at wavenumber intervals of

$$\Delta k = 1/L \quad (5.34)$$

[which correspond to frequency intervals of $\Delta f = 1/T$ — see Eq. (3.13)].

Recognizing that the right-hand side of Eq. (5.31) represents an *average* of $(d/dL) \ln F_j(L)$, and that as T and $L \rightarrow \infty$, our wavenumber spacing $\Delta k \rightarrow 0$, we have for the limiting form of $I^{(1)}$ given by Eq. (5.33a):

$$I^{(1)} = \frac{1}{N \Delta k} \int_{N_1 \Delta k}^{N_2 \Delta k} \frac{d}{dL} \ln F(k; L) dk, \quad (5.35)$$

where

$$N = N_2 - N_1, \quad (5.36)$$

and where we have used the notation of the left-hand identities in Eqs. (3.34) and (3.36). From Eqs. (3.35) and (3.37), we see that $L(d/dL) \ln F(k; L)$ is a function of only the *product* Lk ; hence, let us define

$$H(Lk) \triangleq L \frac{d}{dL} \ln F(k; L). \quad (5.37)$$

Using Eq. (5.37), $I^{(1)}$ can be expressed from Eq. (5.35) as

$$I^{(1)} = \frac{1}{NL^2 \Delta k} \int_{N_1 \Delta k}^{N_2 \Delta k} H(Lk) L dk \quad (5.38)$$

$$= \frac{1}{NL^2 \Delta k} \int_{N_1 L \Delta k}^{N_2 L \Delta k} H(\xi) d\xi, \quad (5.39)$$

where $\xi = Lk$. Let us define

$$H_\infty \triangleq \lim_{\xi \rightarrow \infty} H(\xi). \quad (5.40)$$

Recognizing that we are especially interested in the case of "large" upper limits $N_2 \Delta k$ and $N_2 L \Delta k$ in Eqs. (5.35) and (5.39) respectively, let us now rewrite Eq. (5.39) as

$$\begin{aligned} I^{(1)} &= \frac{1}{NL^2 \Delta k} \int_{N_1 L \Delta k}^{N_2 L \Delta k} [H(\xi) - H_\infty] d\xi + \frac{H_\infty (N_2 - N_1) L \Delta k}{NL^2 \Delta k} \\ &= \frac{1}{NL^2 \Delta k} \int_{N_1 L \Delta k}^{N_2 L \Delta k} [H(\xi) - H_\infty] d\xi + \frac{H_\infty}{L}, \end{aligned} \quad (5.41)$$

where we have used Eq. (5.36).

The likelihood equations (3.25) and (3.26) apply for any limits N_1 and N_2 in Eq. (5.41). However, generally we are interested in a (dimensionless) low wavenumber limit of $k_1 L = N_1 L \Delta k = 0$, whereas the (dimensionless) high wavenumber limit $k_2 L = N_2 L \Delta k$ is usually taken to be the largest wavenumber for which the spectrum is still unaffected by instrumentation errors etc. This highest wavenumber is usually of the order of $k_2 L \approx 10$, where L is the integral scale of the turbulence. From Eqs. (3.35) and (3.37), we see that

$$H(Lk) = LG_j(L) \quad (5.42)$$

which is plotted in Figs. 20 and 21 for the von Karman transverse and longitudinal spectra respectively. We shall show later that for both of these spectra we have

$$H_\infty = -\frac{5}{3} = -1.666\ldots, \quad (5.43)$$

and we see from Figs. 20 and 21 that for values of kL larger than about 3 this asymptotic value has, for practical purposes, been reached. Thus, for cases where $k_1 \approx 0$ and $k_2 L = N_2 L \Delta k \geq 3L$, we may replace Eq. (5.41) by

$$I_\infty^{(1)} \triangleq \frac{1}{NL^2 \Delta k} \int_0^\infty [H(\xi) - H_\infty] d\xi + \frac{H_\infty}{L} = \frac{\gamma^{(1)}}{NL^2 \Delta k} + \frac{H_\infty}{L}, \quad (5.44)$$

where

$$\gamma^{(1)} \triangleq \int_0^\infty [H(\xi) - H_\infty] d\xi. \quad (5.45)$$

Let us turn now to $I^{(2)}$ defined by Eq. (5.32) and its large T asymptotic form defined by Eq. (5.33b) — which is

$$I^{(2)} = \frac{1}{N \Delta k} \int_{N_1 \Delta k}^{N_2 \Delta k} \left[\frac{d}{dL} \ln F(k; L) \right]^2 dk, \quad (5.46)$$

where the above limiting form is arrived at by the same arguments as were used in obtaining Eq. (5.35). Proceeding along the same lines as in the case of $I^{(1)}$, we have upon introduction of the definitions of Eqs. (5.37) and (5.40),

$$\begin{aligned}
I^{(2)} &= \frac{1}{NL^3 \Delta k} \int_{N_1 \Delta k}^{N_2 \Delta k} H^2(Lk) L dk \\
&= \frac{1}{NL^3 \Delta k} \int_{N_1 L \Delta k}^{N_2 L \Delta k} H^2(\xi) d\xi \\
&= \frac{1}{NL^3 \Delta k} \int_{N_1 L \Delta k}^{N_2 L \Delta k} [H^2(\xi) - H_\infty^2] d\xi + \frac{H_\infty^2 (N_2 - N_1) L \Delta k}{NL^3 \Delta k} \\
&= \frac{1}{NL^3 \Delta k} \int_{N_1 L \Delta k}^{N_2 L \Delta k} [H^2(\xi) - H_\infty^2] d\xi + \frac{H_\infty^2}{L^2} , \tag{5.47}
\end{aligned}$$

where $N = N_2 - N_1$ was used in going to the last line. Again recognizing that we are dealing in the integrand with the quantity described by Eq. (5.42), we see from Figs. 20 and 21 that for lower (dimensionless) wavenumber limits $k_1 L = N_1 L \Delta k \approx 0$ and upper limits $k_2 L = N_2 L \Delta k$ larger than about $3L$, we may replace Eq. (5.47) by

$$I_\infty^{(2)} \triangleq \frac{1}{NL^3 \Delta k} \int_0^\infty [H^2(\xi) - H_\infty^2] d\xi + \frac{H_\infty^2}{L^2} = \frac{\gamma^{(2)}}{NL^3 \Delta k} + \frac{H_\infty^2}{L^2} , \tag{5.48}$$

where

$$\gamma^{(2)} \triangleq \int_0^\infty [H^2(\xi) - H_\infty^2] d\xi . \tag{5.49}$$

From Eqs. (5.29), (5.32), and (5.33), it follows that the large T asymptotic form of the correlation coefficient ρ can be expressed as

$$\begin{aligned}
 \rho &= - \frac{I^{(1)}}{(I^{(2)})^{\frac{1}{2}}} \\
 &\approx - \frac{I_{\infty}^{(1)}}{(I_{\infty}^{(2)})^{\frac{1}{2}}} \\
 &= - \frac{H_{\infty} + \frac{\gamma^{(1)}}{NL\Delta k}}{\left(H_{\infty}^2 + \frac{\gamma^{(2)}}{NL\Delta k}\right)^{\frac{1}{2}}} = - \frac{H_{\infty} + \frac{\gamma^{(1)}}{(k_2 - k_1)L}}{\left[H_{\infty}^2 + \frac{\gamma^{(2)}}{(k_2 - k_1)L}\right]^{\frac{1}{2}}}, \quad (5.50)
 \end{aligned}$$

where the second and third lines are valid approximations whenever we have for the lower wavenumber limit in our likelihood equations (3.32) and (3.33) $k_1 \approx 0$, and for the upper wavenumber limit we have $k_2 \geq 3L^{-1}$. Notice from Eq. (5.50) and the fact that H_{∞} is negative that

$$\lim_{k_2 \rightarrow \infty} \rho = 1. \quad (5.51)$$

Hence, when $k_1 \approx 0$ and $k_2 \rightarrow \infty$, our *estimates* of $\sigma^2 L$ and L have a correlation coefficient of unity. This limiting behavior may be understood from the fact that *if L were known exactly*, our estimate of $\sigma^2 L$ given by Eqs. (3.25) or (3.32) would improve indefinitely as $k_2 = N_2 \Delta k$ increases indefinitely; hence, for large enough k_2 all of the statistical uncertainty in our estimate of $\sigma^2 L$ is a consequence of the uncertainty in our estimate of L - which is reflected in the fact that $\rho \rightarrow 1$ as $k_2 \rightarrow \infty$.

Let us turn now to evaluation of $|\Lambda^{-1}|$ given by Eq. (5.30), which is required in our expressions for σ_1^2 and σ_2^2 given by Eqs. (5.27) and (5.28). From Eqs. (5.30) through (5.33), we see that the large T asymptotic form of $|\Lambda^{-1}|/N^2$ can be expressed as

$$\frac{|\Lambda^{-1}|}{N^2} = \frac{1}{(\sigma^2 L)^2} \left[I^{(2)} - \left(I^{(1)} \right)^2 \right] \quad . \quad (5.52)$$

Hence, whenever $k_1 \approx 0$ and $k_2 \geq 3L^{-1}$, we have for practical purposes

$$\begin{aligned} \frac{|\Lambda^{-1}|}{N^2} &= \frac{1}{(\sigma^2 L)^2} \left[I_{\infty}^{(2)} - \left(I_{\infty}^{(1)} \right)^2 \right] \\ &= \frac{1}{L^2 (\sigma^2 L)^2} \left\{ \frac{\gamma^{(2)}}{(k_2 - k_1)L} + H_{\infty}^2 - \left[\frac{\gamma^{(1)}}{(k_2 - k_1)L} + H_{\infty} \right]^2 \right\} \\ &= \frac{1}{L^2 (\sigma^2 L)^2 N L \Delta k} \left[\gamma^{(2)} - 2H_{\infty} \gamma^{(1)} - \frac{(\gamma^{(1)})^2}{(k_2 - k_1)L} \right] \quad , \quad (5.53) \end{aligned}$$

where Eqs. (5.44) and (5.48) and the relationship

$$N \Delta k = (N_2 - N_1) \Delta k = k_2 - k_1 \quad (5.54)$$

have been used in going to the second and third lines in Eq. (5.53). Let us now define

$$\begin{aligned} \gamma^{(3)} &\triangleq \int_0^{\infty} [H(\xi) - H_{\infty}]^2 d\xi \\ &= \int_0^{\infty} [H^2(\xi) - 2H_{\infty}H(\xi) + H_{\infty}^2] d\xi \quad , \\ &= \int_0^{\infty} [H^2(\xi) - H_{\infty}^2] d\xi - 2H_{\infty} \int_0^{\infty} [H(\xi) - H_{\infty}] d\xi \\ &= \gamma^{(2)} - 2H_{\infty} \gamma^{(1)} \quad , \quad (5.55) \end{aligned}$$

according to Eqs. (5.45) and (5.49). Combining Eqs. (5.53) and (5.55) with Eq. (5.34), we find that $|\Lambda^{-1}|/N$ can be expressed as

$$\frac{|\Lambda^{-1}|}{N} = \frac{1}{(\sigma^2 L)^2 L^3 / L} \left[\gamma^{(3)} - \frac{(\gamma^{(1)})^2}{(k_2 - k_1)L} \right], \quad (5.56)$$

where $\gamma^{(3)}$ is expressed in terms of $\gamma^{(1)}$ and $\gamma^{(2)}$ by Eq. (5.55), and where Eq. (5.56) is valid for practical purposes whenever $k_1 \approx 0$ and $k_2 \geq 3L^{-1}$.

If we divide the numerator and denominator of Eq. (5.27) by N , we can express the large T asymptotic form of σ_1^2 as

$$\sigma_1^2 = \frac{I^{(2)}}{|\Lambda^{-1}|/N}, \quad (5.57)$$

where Eqs. (5.32) and (5.33b) have been used. Recalling that σ_1^2 is the variance of our estimate of $\sigma^2 L$, we can express the *relative variance* of our estimate of $\sigma^2 L$ for the case where $k_1 \approx 0$ and $k_2 \geq 3L^{-1}$ using Eqs. (5.48), (5.54), and (5.56) as

$$\begin{aligned} \frac{\sigma_1^2}{(\sigma^2 L)^2} &= \frac{L}{L} \left[\frac{H_\infty^2 + \frac{\gamma^{(2)}}{(k_2 - k_1)L}}{\gamma^{(3)} - \frac{(\gamma^{(1)})^2}{(k_2 - k_1)L}} \right] \\ &= \frac{H_\infty^2}{\gamma^{(3)}} \frac{L}{L} \left[\frac{1 + \frac{\gamma^{(2)}/H_\infty^2}{(k_2 - k_1)L}}{1 - \frac{(\gamma^{(1)})^2/\gamma^{(3)}}{(k_2 - k_1)L}} \right]. \end{aligned} \quad (5.58)$$

Similarly, if we divide the numerator and denominator of Eq. (5.28) by N , we can express the large T asymptotic form of σ_2^2 as

$$\sigma_2^2 = \frac{1}{(\sigma^2 L)^2 |\Lambda^{-1}|/N} . \quad (5.59)$$

Recalling that σ_2^2 is the variance of our estimate of L , we can express the *relative variance* of our estimate of L for the case where $k_1 \approx 0$ and $k_2 \geq 3L^{-1}$, using Eq. (5.56) and (5.59), as

$$\begin{aligned} \frac{\sigma_2^2}{L^2} &= \frac{L}{L} \left[\gamma^{(3)} - \frac{(\gamma^{(1)})^2}{(k_2 - k_1)L} \right]^{-1} \\ &= \frac{1}{\gamma^{(3)}} \frac{L}{L} \left[1 - \frac{(\gamma^{(1)})^2 / \gamma^{(3)}}{(k_2 - k_1)L} \right]^{-1} \end{aligned} \quad (5.60)$$

Finally, recognizing that H_∞ is negative for both the von Karman transverse and longitudinal spectral forms, we can express the correlation coefficient of our maximum likelihood estimates of $\sigma^2 L$ and L from Eq. (5.50) as

$$\rho = \frac{1 + \frac{\gamma^{(1)}/H_\infty}{(k_2 - k_1)L}}{\left[1 + \frac{\gamma^{(2)}/H_\infty^2}{(k_2 - k_1)L} \right]^{\frac{1}{2}}} , \quad (5.61)$$

which is valid whenever $k_1 \approx 0$ and $k_2 \geq 3L^{-1}$.

Equations (5.58), (5.60), and (5.61) are the main results of this section. Each of these three results is the large T asymptotic form of its left-hand side that is valid for practical purposes whenever $k_1 \approx 0$ and $k_2 \geq 3L^{-1}$. It may be seen from Eqs. (3.35), (3.37), (5.37), and (5.40) that for both the von Karman transverse and longitudinal spectra, we have

$$H_\infty = - \frac{117.97}{70.78} = 1.6667 = - \frac{5}{3} , \quad (5.62)$$

where $-5/3$ is the exact asymptotic result. The other dimensionless constants $\gamma^{(1)}$, $\gamma^{(2)}$, and $\gamma^{(3)}$ are defined in terms of the function $H(\xi)$ by Eqs. (5.45), (5.49), and the first line in Eq. (5.55) respectively. The last line in Eq. (5.55) shows that only two of these three constants is independent. The function $H(\xi) = H(Lk)$ is defined by Eq. (5.37), where $(d/dL)\ln F(k;L)$ is given by Eqs. (3.35) and (3.37) for the von Karman transverse and longitudinal spectra respectively. L is the integral scale associated with the von Karman spectral forms, k_1 and k_2 are the lower and upper wavenumber limits used in the likelihood equations — see Eqs. (3.32) and (3.33) — and $L = VT$ is the length of the record.

Coefficient of Variation of Mean-Square Velocity Estimates

We have shown earlier that, as $T \rightarrow \infty$, our maximum likelihood estimates of $\sigma^2 L$ and L are asymptotically governed by a joint normal probability density with mean values equal to the true values of $\sigma^2 L$ and L , and with variances σ_1^2 and σ_2^2 , and correlation coefficient ρ , as given above. In particular, Eq. (5.60) provides a general expression for the variance of our estimate of L (divided by the square of its mean).

To get a comparable expression for the variance of our estimate of σ^2 where

$$\sigma^2 \equiv \frac{\sigma^2 L}{L}, \quad (5.63)$$

we must consider the ratio of our estimates of $\sigma^2 L$ and L as indicated in Eq. (5.63). Since these estimates of $\sigma^2 L$ and L are asymptotically governed by a joint normal probability density, the estimate $\hat{\sigma}^2$ of σ^2 (obtained by dividing an estimate of $\sigma^2 L$ by an estimate of L) is governed by whatever probability density describes the ratio of two (correlated) normal variates each having a nonzero mean value. When both mean values are zero the resulting density of the ratio is a Cauchy probability density — e.g., pp. 153-154 of Ref. 43 — however, when mean values are not zero the resulting density apparently cannot be written in closed form — e.g., p. 411 of Ref. 48. Fortunately though, we may obtain an expression for the variance of our estimate of σ^2 using the method of Ref. 49 — even though its distribution cannot be expressed in closed form.

Merrill (Ref. 49) obtains a series expansion for the variance of the ratio of two normally distributed variables which converges rapidly (at least in the asymptotic sense) whenever the coefficient of variation of the variate in the denominator is small — which in our case occurs whenever σ_1^2/L^2 is small — as will generally be the case. Adapting Merrill's notation to our case, let

$$v_1^2 = \frac{\sigma_2^2}{L^2} \quad v_2^2 = \frac{\sigma_1^2}{(\sigma^2 L)^2} \quad . \quad (5.64 \text{ a,b})$$

Then Merrill's I_0 is our σ^2 and Merrill's I is our *estimate* of σ^2 . At the top of p. 56 of Ref. 49, Merrill gives an expression for the variance of the ratio — i.e., the variance of our estimate of σ^2 — which is

$$\begin{aligned} \text{Var}(\hat{\sigma}^2) = (\sigma^2)^2 \{ & v_1^2 - 2\rho v_1 v_2 + v_2^2 + 8v_1^4 - 16\rho v_1^3 v_2 \\ & + 3v_1^2 v_2^2 + 5\rho^2 v_1^2 v_2^2 + 69v_1^6 - 138\rho v_1^5 v_2 \\ & + 15v_1^4 v_2^2 + 54\rho^2 v_1^4 v_2^2 + \dots \}, \end{aligned} \quad (5.65)$$

where we carried through powers of six in the product v_1 and v_2 .

We are primarily interested in cases where v_1^2 and v_2^2 are small in comparison with unity. From Eqs. (5.58) and (5.60), we see that v_1^2 and v_2^2 are made small by increasing the length L of the turbulence record. In fact, when Eqs. (5.58), (5.60), and (5.61) are substituted into Eq. (5.65) — using Eqs. (5.64 a, b) — we see that the right-hand side of Eq. (5.65) takes on the appearance of a series in powers of L/L . If we retain only the *first* power in L/L , the relative variance of our estimate of σ^2 becomes

$$\frac{\text{Var}(\hat{\sigma}^2)}{(\sigma^2)^2} = v_1^2 - 2\rho v_1 v_2 + v_2^2 + O\left(\frac{L^2}{L^2}\right) \quad , \quad (5.66)$$

where, according to Eqs. (5.64 a,b), v_1^2 and v_2^2 are given by Eqs. (5.60) and (5.58) respectively, and ρ is given by Eq. (5.61). The coefficient of variation of our estimate of σ^2 is $[\text{Var}(\hat{\sigma}^2)/(\sigma^2)^2]^{1/2}$.

The case of most interest is that for which k_2 is very large. To study this case, we can let $(k_2 - k_1)L \rightarrow \infty$. For this limiting case, we see from Eq. (5.51) that $\rho = 1$. Hence, as $(k_1 - k_2)L \rightarrow \infty$, we have from Eq. (5.56) by retaining only terms of first degree in L/L :

$$\begin{aligned} \frac{\text{Var}(\hat{\sigma}^2)}{(\sigma^2)^2} &\sim (v_1 - v_2)^2, & (k_2 - k_1)L \rightarrow \infty \\ &= \left(\frac{\sigma_2}{L} - \frac{\sigma_1}{\sigma^2 L} \right)^2 \\ &\sim \frac{1}{\gamma^{(3)}} (1 - |H_\infty|)^2 \frac{L}{L}, & (k_2 - k_1)L \rightarrow \infty, \end{aligned} \quad (5.67)$$

where we have used Eqs. (5.64 a,b), (5.58), and (5.60) in going to the second and third lines, and where we have retained only the first power of L/L in the last line of Eq. (5.67).

Numerical Results and Discussion

Let us turn now to numerical evaluation of the above described quantities for the von Karman transverse and longitudinal spectra. We shall treat the longitudinal case first since the integrals required for this case are easier to evaluate than those for the lateral case.

The case of most interest is that for which k_2 is very large, so we can let $(k_2 - k_1)L \rightarrow \infty$ as before. In this case, we see from Eqs. (5.58), (5.60), (5.61), and (5.67) that

$\gamma^{(3)}$ is the constant of primary importance — since we have already determined that $H_\infty = -5/3$ [Eq. (5.62)] for both the von Karman transverse and longitudinal spectra.

The expression we shall use for $\gamma^{(3)}$ is that given by the first line in Eq. (5.55):

$$\gamma^{(3)} \triangleq \int_0^\infty [H(\xi) - H_\infty]^2 d\xi, \quad (5.68)$$

where $H(\xi)$ is defined by Eq. (5.37). For the *von Karman longitudinal spectrum*, we have from Eqs. (5.37) and (3.37):

$$H(\xi) = - \frac{117.97\xi^2}{1+70.78\xi^2} . \quad (5.69)$$

Hence, $H(\xi) - H_\infty$ can be expressed as

$$H(\xi) - H_\infty = \frac{117.97}{70.78} - \frac{117.97\xi^2}{1+70.78\xi^2} , \quad (5.70)$$

which is of the form

$$H(\xi) - H_\infty = \frac{a}{b} - \frac{a\xi^2}{1+b\xi^2} = \frac{a}{b(1+b\xi^2)} , \quad (5.71)$$

where

$$\frac{a}{b} = -H_\infty = \frac{5}{3} \quad (5.72)$$

and

$$b = 70.78. \quad (5.73)$$

From Eqs. (5.68) and (5.71) it follows that $\gamma^{(3)}$ can be expressed for the longitudinal von Karman spectrum as

$$\gamma^{(3)} = \left(\frac{a}{b}\right)^2 \int_0^\infty \frac{1}{(1+b\xi^2)^2} d\xi , \quad (5.74)$$

which is of the form of Eq. (3.251.11) on p. 295 of Ref. 50:

$$\int_0^\infty (1+b\xi^2)^{-2} d\xi = \frac{\pi}{4b^{1/2}} . \quad (5.75)$$

Hence, for the von Karman longitudinal spectrum, we have

$$\gamma^{(3)} = \left(\frac{a}{b}\right)^2 \frac{\pi}{4b^{\frac{1}{2}}} = \frac{25}{9} \frac{\pi}{4(70.78)^{\frac{1}{2}}} = 0.2593. \quad (5.76)$$

From Eqs. (5.60) and (5.76), we therefore have for the relative variance of our estimate of L as $(k_2 - k_1)L \rightarrow \infty$,

$$\frac{\sigma_2^2}{L^2} = 3.856 L/L, \quad (5.77)$$

whereas, from Eqs. (5.62), (5.67), and (5.76), we have for the relative variance of our estimate of σ^2 as $(k_2 - k_1)L \rightarrow \infty$,

$$\frac{\text{Var}(\hat{\sigma}^2)}{(\sigma^2)^2} = 1.714 L/L, \quad (5.78)$$

where only the first power of L/L has been retained in Eq. (5.78). Equations (5.77) and (5.78) apply to the von Karman longitudinal spectrum. L is the integral scale of the turbulence and l is the length of the record.

Let us now turn to evaluation of the relative variance of our estimates of L and σ^2 in the case of the von Karman transverse spectrum. For this case, we again require the evaluation of $\gamma^{(3)}$ given by Eq. (5.68) where $H(\xi)$ is defined by Eq. (5.37). For the von Karman transverse spectrum, we have from Eqs. (5.37) and (3.35):

$$H(\xi) = \frac{117.97\xi^2(1-188.75\xi^2)}{(1+70.78\xi^2)(1+188.75\xi^2)}, \quad (5.79)$$

which is of the form

$$H(\xi) = \frac{a\xi^2(1-c\xi^2)}{(1+b\xi^2)(1+c\xi^2)}, \quad (5.80)$$

where

$$\begin{aligned}
 a &= 117.97 \\
 b &= 70.78 = \frac{3}{5} a \\
 c &= 118.75 = \frac{8}{3} b .
 \end{aligned} \tag{5.81}$$

From Eqs. (5.80) and (5.81), we can see that

$$H_{\infty} = \lim_{\xi \rightarrow \infty} H(\xi) = - \frac{ac}{bc} = - \frac{a}{b} = - \frac{5}{3} . \tag{5.82}$$

Therefore,

$$\begin{aligned}
 H(\xi) - H_{\infty} &= \frac{a\xi^2(1-c\xi^2)}{(1+b\xi^2)(1+c\xi^2)} + \frac{a}{b} \\
 &= \frac{a}{b} \left[\frac{(1+b\xi^2)(1+c\xi^2) + b\xi^2(1-c\xi^2)}{(1+b\xi^2)(1+c\xi^2)} \right] \\
 &= \frac{a}{b} \left[\frac{1+(2b+c)\xi^2}{(1+b\xi^2)(1+c\xi^2)} \right].
 \end{aligned} \tag{5.83}$$

Combining Eqs. (5.68) and (5.83) gives us the desired expression for $\gamma^{(3)}$ in the case of the *von Karman transverse spectrum*:

$$\begin{aligned}
 \gamma^{(3)} &= \left(\frac{a}{b} \right)^2 \int_0^{\infty} \left[\frac{1+(2b+c)\xi^2}{(1+b\xi^2)(1+c\xi^2)} \right]^2 d\xi \\
 &= \frac{1}{2} \left(\frac{a}{b} \right)^2 \int_{-\infty}^{\infty} \frac{[1+(2b+c)\xi^2]^2}{(1+b\xi^2)^2(1+c\xi^2)^2} d\xi ,
 \end{aligned} \tag{5.84}$$

since the integrand is an even function of ξ .

The integral on the right-hand side of Eq. (5.84) is readily evaluated by contour integration using the method outlined on pp. 584-587 of Ref. 51. This integration is carried out in Appendix I and yields

$$\gamma(3) = \frac{\pi}{(c-b)^3} \left(\frac{a}{b}\right)^2 \left[\frac{(b+c)(7b^2+c^2)}{4b^{\frac{1}{2}}} - \frac{b(b^2+bc+2c^2)}{c^{\frac{1}{2}}} \right]. \quad (5.85)$$

When the exact values for a , b , and c given by the left-hand sides of Eqs. (3.41), (3.39), and (3.40) respectively [compare Eqs. (3.39) to (3.41) with Eqs. (5.81)] are substituted into Eq. (5.85), we find for the von Karman transverse spectrum that

$$\gamma(3) = 0.4437. \quad (5.86)$$

From Eqs. (5.60) and (5.86), we therefore have for the relative variance of our estimate of L as $(k_2-k_1)L \rightarrow \infty$,

$$\frac{\sigma^2}{L^2} = 2.254L/L, \quad (5.87)$$

whereas, from Eqs. (5.67), (5.82), and (5.86), we have for the relative variance of our estimate of σ^2 as $(k_2-k_1)L \rightarrow \infty$,

$$\frac{\text{Var}(\hat{\sigma}^2)}{(\sigma^2)^2} = 1.002L/L, \quad (5.88)$$

where we have again retained only the first power in L/L in Eq. (5.88). Equations (5.87) and (5.88) apply to the von Karman transverse spectrum. L is the integral scale of the turbulence and L is the length of the record.

Equations (5.77) and (5.78), which apply to the von Karman longitudinal spectrum, and Eqs. (5.87) and (5.88), which apply to the von Karman transverse spectrum, are the main numerical results of this section.

Discussion. It is instructive to compare the values of relative variance given by Eqs. (5.78) and (5.88), which apply to the von Karman longitudinal and transverse spectra respectively, with the values of relative variance of estimates of σ^2 obtained by

squaring and averaging velocity records. The relative variances obtained for this latter estimation procedure are 1.732 L/L and 1.069 L/L [Eqs. (D.28) and (3.21) of Ref. 18] for von Karman longitudinal and transverse records respectively. Hence, the relative variances given by Eqs. (5.78) and (5.88) of maximum likelihood estimates of σ^2 — i.e., 1.714 L/L and 1.002 L/L — are only very slightly smaller than the values of 1.732 L/L and 1.069 L/L for the squaring and averaging estimation procedure. Nevertheless, the maximum likelihood method relative variances are smaller, as we would expect from the asymptotic efficiency normally associated with maximum likelihood estimates.

The relative variances of 3.856 L/L and 2.254 L/L for maximum likelihood estimates of L for von Karman longitudinal and transverse records respectively are, perhaps, of more interest. In contrast to the squaring and averaging procedure used to estimate σ^2 , reliable estimation procedures and associated variances for obtaining the integral scale from velocity records have not, in the past, existed.

AIRCRAFT RESPONSE EXCEEDANCE RATES

In Secs. 4.2 and 4.3 of Ref. 19, a series expansion was developed for the mean rate of exceedances $N_+(y)$ *with positive slope* of a generic aircraft response variable past a specified level y . This result was derived for aircraft responses to the three component turbulence model of Eqs. (1.2) to (1.4) of the present report. However, these results are valid only for situations where the three locally stationary conditions described in Ref. 19 are satisfied. In this section, a new expansion for the mean rate of exceedances $N_+(y)$ is developed; this new result depends only on the validity of the first locally stationary condition described by Eq. (1.8). This condition depends on turbulence properties only, and is believed to be virtually always satisfied. Thus, the results derived herein apply to supersonic aircraft with arbitrarily high Mach numbers — as well as to subsonic aircraft for which the results of Secs. 4.2 and 4.3 of Ref. 19 should apply. However, the derivations contained in the present section are considerably more involved than those of Secs. 4.2 and 4.3 of Ref. 19, and in order to hold the complexity within bounds, *we have assumed in the present section that the slow turbulence component $w_s(t)$ in Eq. (1.2) is negligible in comparison with the fast component $w_f(t)$.*

Application of Rice's Formula to Intensity Modulated Gaussian Processes

In the derivation to follow, we shall evaluate Rice's expression for the mean number of crossings with positive slope per unit time $N_+(y)$ of a stationary process past the level y . It was shown by Rice [5] on p. 189-193 of the Wax edition that, for stationary processes, one has

$$N_+(y) = \int_0^{\infty} \dot{y} p(y, \dot{y}) d\dot{y}, \quad (6.1)$$

where $p(y, \dot{y})$ is the joint probability density function of the aircraft response y and its time derivative \dot{y} . A derivation of Eq. (6.1) also can be found on pp. 45-47 of Crandall and Mark [29].

To evaluate $N_+(y)$ from Eq. (6.1), we shall use an extension of the methodology developed in Sec. 2 of this report for evaluating the first-order probability density of the process $y(t)$. On the first two pages of Sec. 2, we showed that the aircraft response $y(t)$ conditioned on the process $\sigma_f(u)$ for all $-\infty < u \leq t$ is strictly Gaussian with zero mean value, but generally nonstationary. This conditional response process is denoted by $\{y(t)|\sigma_f(u)\}$, $-\infty < u \leq t$. Therefore, the joint probability density function of $y(t)$ and $\dot{y}(t)$ conditioned on $\sigma_f(u)$ for $-\infty < u \leq t$ is a joint normal density function with zero mean values for $y(t)$ and $\dot{y}(t)$. Let us denote this joint conditional density function by $p(y, \dot{y}|\underline{g}_f)$, where \underline{g}_f denotes the infinite dimensional "vector" $\sigma_f(u)$, $-\infty < u \leq t$, as described on the third page in Sec. 2 of this report. We therefore may formally express the unconditional joint probability density of the aircraft response and its first derivative as

$$p(y, \dot{y}) = \int_{\underline{0}}^{\infty} p(y, \dot{y}|\underline{g}_f) p(\underline{g}_f) d\underline{g}_f, \quad (6.2)$$

where $p(\underline{g}_f)$ denotes the probability density of the infinite dimensional vector $\underline{g}_f \equiv \sigma_f(u)$, $-\infty < u \leq t$, and where the symbolic integration is taken over this same infinite dimensional space as described on the third and fourth pages of Sec. 2 of this report. Substitution of Eq. (6.2) into Eq. (6.1) yields

$$N_+(y) = \int_0^{\infty} \dot{y} \int_{\underline{0}}^{\infty} p(y, \dot{y}|\underline{g}_f) p(\underline{g}_f) d\underline{g}_f d\dot{y} \quad (6.3)$$

When $\underline{g}_f \equiv \sigma_f(u)$, $-\infty < u \leq t$ is specified, the joint density of y and \dot{y} is a joint normal density with zero mean values in y and \dot{y} [Ref. 39, pp. 147, 148] - i.e.,

$$\begin{aligned} p(y, \dot{y}|\underline{g}_f) &\equiv p(y, \dot{y}|\sigma_y^2, \sigma_{\dot{y}}^2, \mu_{y\dot{y}}) \\ &= \frac{1}{2\pi(\sigma_y^2\sigma_{\dot{y}}^2 - \mu_{y\dot{y}}^2)^{\frac{1}{2}}} e^{-\frac{(\sigma_{\dot{y}}^2 y^2 - 2\mu_{y\dot{y}} y\dot{y} + \sigma_y^2 \dot{y}^2)}{2(\sigma_y^2\sigma_{\dot{y}}^2 - \mu_{y\dot{y}}^2)}} \end{aligned} \quad (6.4)$$

where we have defined the conditional expectations σ_y^2 , $\sigma_{\dot{y}}^2$, and $\mu_{y\dot{y}}$ as

$$\sigma_y^2 \equiv \sigma_y^2(t) \triangleq E_z\{y^2(t)|\sigma_f(u)\}, \quad -\infty < u \leq t \quad (6.5a)$$

$$\sigma_{\dot{y}}^2 \equiv \sigma_{\dot{y}}^2(t) \triangleq E_z\{\dot{y}^2(t)|\sigma_f(u)\}, \quad -\infty < u \leq t \quad (6.5b)$$

$$\mu_{y\dot{y}} \equiv \mu_{y\dot{y}}(t) \triangleq E_z\{y(t)\dot{y}(t)|\sigma_f(u)\}, \quad -\infty < u \leq t, \quad (6.5c)$$

each of which is a stochastic function of time t that is determined by the behavior of the stochastic modulating process $\sigma_f(u)$ over the time interval $-\infty < u \leq t$. Thus, the expectation operations in Eqs. (6.5a) to (6.5c) are taken over variations in the (stationary Gaussian) modulated process $\{z(t)\}$ in our turbulence model of Eqs. (1.2) to (1.4). [Recall that, in this section, we are assuming $w_s(t) \equiv 0$.] We have indicated that these expectations are with respect to the process $\{z(t)\}$ by placing the subscript "z" after the expectation operator "E" in Eqs. (6.5a) to (6.5c).

Noting that Eq. (6.2) expresses the mathematical expectation of the response joint conditional density $p(y, \dot{y}|\underline{g}_f)$ with respect to the modulating process $\{\sigma_f(u)\}$, $-\infty < u \leq t$, we can also express Eq. (6.2) as the expectation

$$p(y, \dot{y}) = E_{\sigma_f}\{p(y, \dot{y}|\underline{g}_f)\}, \quad (6.6)$$

which when combined with Eq. (6.1) yields

$$N_+(y) = \int_0^\infty \dot{y} E_{\sigma_f}\{p(y, \dot{y}|\underline{g}_f)\} d\dot{y}, \quad (6.7)$$

which we shall now proceed to evaluate in terms of measurable metrics of the modulating process $\{\sigma_f(t)\}$.

Series Expansion of Conditional Joint Probability Density of Aircraft Response Displacement and Its Derivative

As in Sec. 2, we are interested here in excitation processes $w_f(t) = \sigma_f(t) z(t)$ where *typical* fluctuations in $\sigma_f(t)$ are not more than about one-third* of the mean value of $\sigma_f(t)$. Different sample functions $\sigma_f(u)$, $-\infty < u \leq t$ give rise to different values of the *three* parameters $\overline{\sigma_y^2}(t)$, $\sigma_{\dot{y}}^2(t)$, and $\mu_{y\dot{y}}(t)$ in the probability density, Eq. (6.4), as may be seen from Eqs. (6.5a) to (6.5c). Hence, in a manner analogous to the approach used in Sec. 2, we require here a truncated Taylor's series representation of the joint normal density, Eq. (6.4), in the three-dimensional parameter space σ_y^2 , $\sigma_{\dot{y}}^2$, and $\mu_{y\dot{y}}$. The expansion will be centered about the point defined by the mean values $\overline{\sigma_y^2}$, $\overline{\sigma_{\dot{y}}^2}$, and $\overline{\mu_{y\dot{y}}}$ of the variables $\sigma_y^2(t)$, $\sigma_{\dot{y}}^2(t)$, and $\mu_{y\dot{y}}(t)$:

*The approximate upper limit of one-third for the *typical* fluctuation $\delta\sigma_f(t)$ relative to the mean σ_f is arrived at as follows. Denoting expected values by overbars and fluctuations by delta, we have

$$\sigma = \bar{\sigma} + \delta\sigma, \quad (a)$$

hence

$$\sigma^2 = (\bar{\sigma})^2 + 2\bar{\sigma}\delta\sigma + (\delta\sigma)^2, \quad (b)$$

and

$$\overline{\sigma^2} = (\bar{\sigma})^2 + \overline{(\delta\sigma)^2}, \quad (c)$$

since $\overline{\delta\sigma} \equiv 0$ by definition. Therefore, from (b) and (c), we have

$$\begin{aligned} \sigma^2 - \overline{\sigma^2} &= 2\bar{\sigma}\delta\sigma + (\delta\sigma)^2 - \overline{(\delta\sigma)^2} \\ &\approx 2\bar{\sigma}\delta\sigma, \end{aligned} \quad (d)$$

since, by assumption, we have $\delta\sigma \ll \bar{\sigma}$. From (d) it follows that

$$\overline{\sigma_y^2} \triangleq E_{\sigma_f} \{ \sigma_y^2(t) \} \quad (6.8a)$$

$$\overline{\sigma_{\dot{y}}^2} \triangleq E_{\sigma_f} \{ \sigma_{\dot{y}}^2(t) \} \quad (6.8b)$$

and

$$\overline{\mu_{y\dot{y}}} \triangleq E_{\sigma_f} \{ \mu_{y\dot{y}}(t) \}, \quad (6.8c)$$

where we have again used the subscript on the expectation operator to denote the process that the expectation is taken with respect to.

$$E\{(\sigma^2 - \overline{\sigma^2})^2\} \approx 4(\overline{\sigma})^2 E\{(\delta\sigma)^2\}. \quad (e)$$

From (c), we also have when $\delta\sigma \ll \overline{\sigma}$

$$\overline{\sigma^2} \approx (\overline{\sigma})^2; \quad (f)$$

hence, combining (e) and (f) yields

$$\frac{E\{(\sigma^2 - \overline{\sigma^2})^2\}}{(\overline{\sigma^2})^2} \approx 4 \frac{E\{(\delta\sigma)^2\}}{(\overline{\sigma})^2}, \quad (g)$$

where $E\{(\delta\sigma)^2\}$ is the variance of $\delta\sigma$ since $\overline{\delta\sigma} \equiv 0$. Applying (g) to σ_y , and using the definition of γ given by Eq. (2.27a), we have

$$\left[\frac{E\{(\delta\sigma_y)^2\}}{(\overline{\sigma_y})^2} \right]^{1/2} \approx \left[\frac{E\{(\sigma_y^2 - \overline{\sigma_y^2})^2\}}{4(\overline{\sigma_y^2})^2} \right]^{1/2} \equiv \frac{1}{\sqrt{4\gamma}}. \quad (h)$$

Moreover, from Figs. 9 to 16, we see that the largest value of γ that gives "good" accuracy is $\gamma = 2$. For this value, we have $(4\gamma)^{-1/2} \equiv 8^{-1/2} = 0.354$ which is slightly more than one-third. Therefore, since typical relative fluctuations in $\sigma_y(t)$ should never be larger than those of $\sigma_f(t)$, we conclude that whenever typical fluctuations in $\sigma_f(t)$ are not larger than one-third of the mean value of $\sigma_f(t)$, our methods should provide reasonably accurate results.

In order to evaluate a common form of the multi-dimensional Taylor's series [e.g., p. 338 of Ref. 35], we require an expression for the square of a trinomial:

$$\begin{aligned}(x+y+z)^2 &= (x+y)^2 + 2(x+y)z + z^2 \\ &= x^2 + y^2 + z^2 + 2xy + 2xz + 2yz.\end{aligned}\tag{6.9}$$

By combining Eq. (6.9) with Eq. (57) on p. 338 of Ref. 35, we obtain a Taylor's series representation of the right-hand side of Eq. (6.4) about the point $\sigma_y^2, \sigma_{\dot{y}}^2, \overline{\mu_{y\dot{y}}}$. Taking the expected value of this representation with respect to the process $\sigma_f(t)$ yields in straightforward fashion

$$\begin{aligned}E_{\sigma_f}\{p(y, \dot{y} | \sigma_f)\} &= p(y, \dot{y} | \overline{\sigma_y^2}, \overline{\sigma_{\dot{y}}^2}, \overline{\mu_{y\dot{y}}}) \\ &+ E_{\sigma_f}\{(\sigma_y^2 - \overline{\sigma_y^2})\} p^{(1,0,0)}(y, \dot{y} | \overline{\sigma_y^2}, \overline{\sigma_{\dot{y}}^2}, \overline{\mu_{y\dot{y}}}) \\ &+ E_{\sigma_f}\{(\sigma_{\dot{y}}^2 - \overline{\sigma_{\dot{y}}^2})\} p^{(0,1,0)}(y, \dot{y} | \overline{\sigma_y^2}, \overline{\sigma_{\dot{y}}^2}, \overline{\mu_{y\dot{y}}}) \\ &+ E_{\sigma_f}\{(\mu_{y\dot{y}} - \overline{\mu_{y\dot{y}}})\} p^{(0,0,1)}(y, \dot{y} | \overline{\sigma_y^2}, \overline{\sigma_{\dot{y}}^2}, \overline{\mu_{y\dot{y}}}) \\ &+ \frac{1}{2} E_{\sigma_f}\{(\sigma_y^2 - \overline{\sigma_y^2})^2\} p^{(2,0,0)}(y, \dot{y} | \overline{\sigma_y^2}, \overline{\sigma_{\dot{y}}^2}, \overline{\mu_{y\dot{y}}}) \\ &+ \frac{1}{2} E_{\sigma_f}\{(\sigma_{\dot{y}}^2 - \overline{\sigma_{\dot{y}}^2})^2\} p^{(0,2,0)}(y, \dot{y} | \overline{\sigma_y^2}, \overline{\sigma_{\dot{y}}^2}, \overline{\mu_{y\dot{y}}}) \\ &+ \frac{1}{2} E_{\sigma_f}\{(\mu_{y\dot{y}} - \overline{\mu_{y\dot{y}}})^2\} p^{(0,0,2)}(y, \dot{y} | \overline{\sigma_y^2}, \overline{\sigma_{\dot{y}}^2}, \overline{\mu_{y\dot{y}}}) \\ &+ E_{\sigma_f}\{(\sigma_y^2 - \overline{\sigma_y^2})(\sigma_{\dot{y}}^2 - \overline{\sigma_{\dot{y}}^2})\} p^{(1,1,0)}(y, \dot{y} | \overline{\sigma_y^2}, \overline{\sigma_{\dot{y}}^2}, \overline{\mu_{y\dot{y}}}) \\ &+ E_{\sigma_f}\{(\sigma_y^2 - \overline{\sigma_y^2})(\mu_{y\dot{y}} - \overline{\mu_{y\dot{y}}})\} p^{(1,0,1)}(y, \dot{y} | \overline{\sigma_y^2}, \overline{\sigma_{\dot{y}}^2}, \overline{\mu_{y\dot{y}}}) \\ &+ E_{\sigma_f}\{(\sigma_{\dot{y}}^2 - \overline{\sigma_{\dot{y}}^2})(\mu_{y\dot{y}} - \overline{\mu_{y\dot{y}}})\} p^{(0,1,1)}(y, \dot{y} | \overline{\sigma_y^2}, \overline{\sigma_{\dot{y}}^2}, \overline{\mu_{y\dot{y}}}) \\ &+ \text{higher order terms},\end{aligned}\tag{6.10}$$

where Eq. (6.9) was used in writing out the last six terms in Eq. (6.10). In Eq. (6.10), we have used the definition

$$p^{(i,j,k)}(y, \dot{y} | \overline{\sigma_y^2}, \overline{\sigma_{\dot{y}}^2}, \overline{\mu_{y\dot{y}}})$$

$$\triangleq \frac{\partial^{i+j+k}}{\partial (\sigma_y^2)^i \partial (\sigma_{\dot{y}}^2)^j \partial (\mu_{y\dot{y}})^k} p(y, \dot{y} | \sigma_y^2, \sigma_{\dot{y}}^2, \mu_{y\dot{y}}) \bigg|_{\substack{\sigma_y^2 = \overline{\sigma_y^2} \\ \sigma_{\dot{y}}^2 = \overline{\sigma_{\dot{y}}^2} \\ \mu_{y\dot{y}} = \overline{\mu_{y\dot{y}}}}}$$
(6.11)

where $p(y, \dot{y} | \sigma_y^2, \sigma_{\dot{y}}^2, \mu_{y\dot{y}})$ is the joint normal density given by the right-hand side of Eq. (6.4), and where the partial derivatives are evaluated at the expansion center defined by Eqs. (6.8a) to (6.8c). The partial derivatives evaluated at this expansion center are not random variables. Thus, the expectation operation with respect to the process $\sigma_f(t)$, $E_{\sigma_f}\{p(y, \dot{y} | \sigma_f)\}$, in the left-hand side of Eq. (6.10) yields the expectations $E_{\sigma_f}\{\dots\}$ of the various expansion coefficients

in the right-hand side of Eq. (6.10). In Eq. (6.10), we have included all terms with partial derivatives of order $i+j+k = 2$ or less, which contain all terms with powers of two or less in products of the random variables $(\sigma_y^2 - \overline{\sigma_y^2})$, $(\sigma_{\dot{y}}^2 - \overline{\sigma_{\dot{y}}^2})$, $(\mu_{y\dot{y}} - \overline{\mu_{y\dot{y}}})$. It is an implicit property of the Taylor's series expansion that the terms written out in the right-hand side of Eq. (6.10) are the most significant terms — provided that fluctuations in $\sigma_f(t)$ about its mean values are not too large as indicated earlier. Since, we have

$$E_{\sigma_f}\{(\sigma_y^2 - \overline{\sigma_y^2})\} = 0$$

$$E_{\sigma_f}\{(\sigma_{\dot{y}}^2 - \overline{\sigma_{\dot{y}}^2})\} = 0$$

and

$$E_{\sigma_f}\{(\mu_{y\dot{y}} - \overline{\mu_{y\dot{y}}})\} = 0, \quad (6.12)$$

the first three "correction" terms to $p(y, \dot{y} | \overline{\sigma_y^2}, \overline{\sigma_{\dot{y}}^2}, \overline{\mu_{y\dot{y}}})$ in the right-hand side of Eq. (6.10) are identically zero. We shall now proceed to develop expressions for the six remaining correction terms written out in Eq. (6.10).

Evaluation of derivatives of joint Gaussian probability density. The somewhat tedious job of evaluating the six partial derivatives $p^{(i,j,k)}(y, \dot{y} | \sigma_y^2, \sigma_{\dot{y}}^2, \mu_{y\dot{y}})$, $i+j+k = 2$, required in Eq. (6.10) is carried out in Appendix J of this report. The resulting expressions evaluated at the expected values of σ_y^2 , $\sigma_{\dot{y}}^2$, and $\mu_{y\dot{y}}$ for use in Eq. (6.10) are given by Eqs. (6.15) to (6.20). In evaluating these expressions we have used the fact that

$$\overline{\mu_{y\dot{y}}} \equiv E_{\sigma_f} \{ \mu_{y\dot{y}} \} = 0 , \quad (6.13)$$

which is shown later by Eq. (6.74). Equation (6.14) gives the joint density of y and \dot{y} also evaluated at the expected values of σ_y^2 , $\sigma_{\dot{y}}^2$, and $\mu_{y\dot{y}}$:

$$p(y, \dot{y} | \overline{\sigma_y^2}, \overline{\sigma_{\dot{y}}^2}, \overline{\mu_{y\dot{y}}}) = \frac{1}{2\pi(\overline{\sigma_y^2} \overline{\sigma_{\dot{y}}^2})^{1/2}} e^{-\left(\frac{y^2}{2\overline{\sigma_y^2}} + \frac{\dot{y}^2}{2\overline{\sigma_{\dot{y}}^2}}\right)} \quad (6.14)$$

$$p^{(2,0,0)}(y, \dot{y} | \overline{\sigma_y^2}, \overline{\sigma_{\dot{y}}^2}, \overline{\mu_{y\dot{y}}}) = \frac{p}{4(\overline{\sigma_y^2})^2} \left[3 - 6 \frac{y^2}{\overline{\sigma_y^2}} + \frac{y^4}{(\overline{\sigma_y^2})^2} \right] \quad (6.15)$$

$$p^{(0,2,0)}(y, \dot{y} | \overline{\sigma_y^2}, \overline{\sigma_{\dot{y}}^2}, \overline{\mu_{y\dot{y}}}) = \frac{p}{4(\overline{\sigma_{\dot{y}}^2})^2} \left[3 - 6 \frac{\dot{y}^2}{\overline{\sigma_{\dot{y}}^2}} + \frac{\dot{y}^4}{(\overline{\sigma_{\dot{y}}^2})^2} \right] \quad (6.16)$$

$$p^{(0,0,2)}(y, \dot{y} | \overline{\sigma_y^2}, \overline{\sigma_{\dot{y}}^2}, \overline{\mu_{y\dot{y}}}) = \frac{p}{\overline{\sigma_y^2} \overline{\sigma_{\dot{y}}^2}} \left(1 - \frac{y^2}{\overline{\sigma_y^2}} \right) \left(1 - \frac{\dot{y}^2}{\overline{\sigma_{\dot{y}}^2}} \right) \quad (6.17)$$

$$p^{(1,1,0)}(y, \dot{y} | \overline{\sigma_y^2}, \overline{\sigma_{\dot{y}}^2}, \overline{\mu_{y\dot{y}}}) = \frac{p}{4\overline{\sigma_y^2} \overline{\sigma_{\dot{y}}^2}} \left(1 - \frac{y^2}{\overline{\sigma_y^2}}\right) \left(1 - \frac{\dot{y}^2}{\overline{\sigma_{\dot{y}}^2}}\right) \quad (6.18)$$

$$\begin{aligned} p^{(1,0,1)}(y, \dot{y} | \overline{\sigma_y^2}, \overline{\sigma_{\dot{y}}^2}, \overline{\mu_{y\dot{y}}}) &= \\ &= \frac{p}{2(\overline{\sigma_y^2})^{3/2} (\overline{\sigma_{\dot{y}}^2})^{1/2}} \frac{y}{(\overline{\sigma_y^2})^{1/2}} \left(\frac{y^2}{\overline{\sigma_y^2}} - 3\right) \frac{\dot{y}}{(\overline{\sigma_{\dot{y}}^2})^{1/2}} \end{aligned} \quad (6.19)$$

$$\begin{aligned} p^{(0,1,1)}(y, \dot{y} | \overline{\sigma_y^2}, \overline{\sigma_{\dot{y}}^2}, \overline{\mu_{y\dot{y}}}) &= \\ &= \frac{p}{2(\overline{\sigma_y^2})^{1/2} (\overline{\sigma_{\dot{y}}^2})^{3/2}} \frac{y}{(\overline{\sigma_y^2})^{1/2}} \left[\frac{\dot{y}^3}{(\overline{\sigma_{\dot{y}}^2})^{3/2}} - 3 \frac{\dot{y}}{(\overline{\sigma_{\dot{y}}^2})^{1/2}} \right] \end{aligned} \quad (6.20)$$

In each of the above six relationships, p is the joint normal density function given by Eq. (6.14); furthermore, in arriving at each of these six relationships, we have used the fact that $\overline{\mu_{y\dot{y}}} = 0$ as indicated by Eq. (6.13).

Series Expansion for Aircraft Response Exceedance Rates

According to Eqs. (6.7) and (6.10), we must now multiply each of the above seven expressions, Eqs. (6.14) to (6.20), by \dot{y} and integrate each resulting expression with respect to \dot{y} from zero to infinity. Examining Eqs. (6.14) to (6.20), we see that these integrations require evaluation in the following expression for values of n from 1 to 5:

$$I(n) \triangleq \int_0^\infty \dot{y}^n p(y, \dot{y} | \overline{\sigma_y^2}, \overline{\sigma_{\dot{y}}^2}, \overline{\mu_{y\dot{y}}}) d\dot{y} \quad (6.21)$$

These integrals are easily evaluated with the aid of Eqs. (3.461-2) and (3.461-3) on pp. 337 of Ref. 50 yielding:

$$I(n) = \frac{1}{2\pi} \left(\frac{\overline{\sigma_y^2}}{\sigma_y^2} \right)^{\frac{1}{2}} e^{-\frac{y^2}{2\overline{\sigma_y^2}}} \left(\frac{n-1}{2} \right)! \left(2 \overline{\sigma_y^2} \right)^{\frac{n-1}{2}},$$

$$n = 1, 3, 5, \dots \quad (6.22)$$

and

$$I(n) = \frac{1}{2\pi} \left(\frac{\overline{\sigma_y^2}}{\sigma_y^2} \right)^{\frac{1}{2}} e^{-\frac{y^2}{2\overline{\sigma_y^2}}} \left(\frac{\pi}{2} \right)^{\frac{1}{2}} (n-1)!! \left(\overline{\sigma_y^2} \right)^{\frac{n-1}{2}},$$

$$n = 2, 4, 6, \dots, \quad (6.23)$$

where in Eq. (6.23), we have used the definition

$$(n-1)!! \triangleq 1 \cdot 3 \cdot 5 \cdots (n-1), \quad (6.24)$$

and in Eq. (6.22) we have used the usual definition of a factorial. Evaluation of Eqs. (6.23) and (6.24) for n from 1 to 5 gives

$$I(1) = \frac{1}{2\pi} \left(\frac{\overline{\sigma_y^2}}{\sigma_y^2} \right)^{\frac{1}{2}} e^{-\frac{y^2}{2\overline{\sigma_y^2}}} \quad (6.25)$$

$$I(2) = \left(\frac{\pi}{2} \right)^{\frac{1}{2}} \left(\overline{\sigma_y^2} \right)^{\frac{1}{2}} I(1) \quad (6.26)$$

$$I(3) = 2 \overline{\sigma_y^2} I(1) \quad (6.27)$$

$$I(4) = 3 \left(\frac{\pi}{2} \right)^{\frac{1}{2}} \left(\overline{\sigma_y^2} \right)^{\frac{3}{2}} I(1) \quad (6.28)$$

and

$$I(5) = 8 \left(\overline{\sigma_y^2} \right)^2 I(1) . \quad (6.29)$$

When Eq. (6.10) is substituted into Eq. (6.7), we see that we must multiply each of Eqs. (6.14) through (6.20) by \dot{y} and integrate the resulting expressions from zero to infinity. When these integrations are carried out with the aid of Eqs. (6.25) through (6.29), we find that the mean threshold crossing rate with positive slope $N_+(y)$ can be expressed as

$$\begin{aligned}
N_+(y) = & \frac{1}{2\pi} \left(\frac{\overline{\sigma_y^2}}{\overline{\sigma_{\dot{y}}^2}} \right)^{\frac{1}{2}} e^{-\frac{y^2}{2\overline{\sigma_y^2}}} \left\{ 1 \right. \\
& + \frac{1}{8} \frac{E_{\sigma_f} \{ (\overline{\sigma_y^2} - \overline{\sigma_{\dot{y}}^2})^2 \}}{(\overline{\sigma_y^2})^2} \left[3 - 6 \frac{y^2}{\overline{\sigma_y^2}} + \frac{y^4}{(\overline{\sigma_y^2})^2} \right] \\
& - \frac{1}{8} \frac{E_{\sigma_f} \{ (\overline{\sigma_{\dot{y}}^2} - \overline{\sigma_y^2})^2 \}}{(\overline{\sigma_{\dot{y}}^2})^2} \\
& - \frac{1}{2} \frac{E_{\sigma_f} \{ (\overline{\mu_{y\dot{y}}} - \overline{\mu_{y\dot{y}}})^2 \}}{\overline{\sigma_y^2} \overline{\sigma_{\dot{y}}^2}} \left(1 - \frac{y^2}{\overline{\sigma_y^2}} \right) \\
& - \frac{1}{4} \frac{E_{\sigma_f} \{ (\overline{\sigma_y^2} - \overline{\sigma_{\dot{y}}^2}) (\overline{\sigma_{\dot{y}}^2} - \overline{\sigma_y^2}) \}}{\overline{\sigma_y^2} \overline{\sigma_{\dot{y}}^2}} \left(1 - \frac{y^2}{\overline{\sigma_y^2}} \right) \\
& - \left(\frac{\pi}{8} \right)^{\frac{1}{2}} \frac{E_{\sigma_f} \{ (\overline{\sigma_y^2} - \overline{\sigma_{\dot{y}}^2}) (\overline{\mu_{y\dot{y}}} - \overline{\mu_{y\dot{y}}}) \}}{(\overline{\sigma_y^2})^{3/2} (\overline{\sigma_{\dot{y}}^2})^{1/2}} \left[3 \frac{y}{(\overline{\sigma_y^2})^{1/2}} - \frac{y^3}{(\overline{\sigma_y^2})^{3/2}} \right] \Big\} \\
& + \text{higher order terms.}
\end{aligned} \tag{6.30}$$

In arriving at Eq. (6.30), we have made use of Eq. (6.12) and the fact that

$$\int_0^{\infty} \dot{y} p^{(0,1,1)}(y, \dot{y} | \overline{\sigma_y^2}, \overline{\sigma_{\dot{y}}^2}, \overline{\mu_{y\dot{y}}}) d\dot{y} = 0. \quad (6.31)$$

The "higher order terms" in Eq. (6.30) arise from the terms with the same label in Eq. (6.10). As mentioned earlier, the "slow" turbulence component $w_s(t)$ in the turbulence model of Eq. (1.2) has been assumed to be zero in arriving at Eq. (6.30). We also have used the fact that $\overline{\mu_{y\dot{y}}} \equiv 0$ in arriving at Eq. (6.30). *Later on in this section, we shall show that the last displayed term in Eq. (6.30) is identically zero, [Eq. (6.71)].*

The problem of finding expressions for the expansion coefficients in Eq. (6.30) will be addressed next.

Expression for Exceedance Rate Expansion Coefficients

Here, we shall derive a general expression for the expansion coefficients $E_{\sigma_f}\{\dots\}$ in Eq. (6.30). We shall then show how the various individual expansion coefficients in Eq. (6.30) can be computed for an arbitrary aircraft modeled as a linear time-invariant system.

All three conditional response variables defined by Eqs. (6.5a) to (6.5c) can be expressed by the single quantity

$$\mu_{jk} \equiv \mu_{jk}(t) \triangleq E_z\{y^j(t)\dot{y}^k(t) | \sigma_f(u)\}, \quad -\infty < u \leq t, \\ j + k = 2, \quad (6.32)$$

where superscripts j and k denote powers of $y(t)$ and $\dot{y}(t)$ respectively, and $E_z\{\dots | \sigma_f(u)\}$, $-\infty < u \leq t$ denotes expectation with respect to the process $\{z(t)\}$ conditioned on the process $\{\sigma_f(u)\}$ for $-\infty < u \leq t$, where $z(t)$ and $\sigma_f(t)$ are components in our turbulence model of Eqs. (1.2) to (1.4). Thus, by comparing Eqs. (6.5a) through (6.5c) with Eq. (6.32), we see that

$$\sigma_y^2 \equiv \mu_{jk} \quad \text{for } j = 2, k = 0 \quad (6.33a)$$

$$\sigma_{\dot{y}}^2 \equiv \mu_{jk} \quad \text{for } j = 0, k = 2 \quad (6.33b)$$

$$\mu_{y\dot{y}} \equiv \mu_{jk} \quad \text{for } j = 1, k = 1 \quad (6.33c)$$

Using the notation of Eqs. (6.32) and (6.33), we can now express any of the expansion coefficients in Eq. (6.30) in the general form

$$\begin{aligned} & E_{\sigma_f} \{ (\mu_{j',k'} - \bar{\mu}_{j',k'}) (\mu_{j'',k''} - \bar{\mu}_{j'',k''}) \} \\ & \triangleq E_{\sigma_f} \left\{ \left(E_Z [y^{j'}(t) \dot{y}^{k'}(t) | \underline{\sigma}_f] - \overline{E_Z [y^{j'}(t) \dot{y}^{k'}(t) | \underline{\sigma}_f]} \right) \right. \\ & \times \left. \left(E_Z [y^{j''}(t) \dot{y}^{k''}(t) | \underline{\sigma}_f] - \overline{E_Z [y^{j''}(t) \dot{y}^{k''}(t) | \underline{\sigma}_f]} \right) \right\}, \quad (6.34) \end{aligned}$$

where we have used $\underline{\sigma}_f$ to denote conditioning on $\sigma_f(u)$, $-\infty < u \leq t$ as before, and where the overbar denotes an expectation with respect to the process $\{\sigma_f(t)\}$. The individual expansion coefficients in Eq. (6.30) that are particular cases of Eq. (6.34) are obtained using the values of j' , k' , j'' , and k'' shown in Table 1. The last entry, $E_{\sigma_f} \{ (\sigma_{\dot{y}}^2 - \overline{\sigma_{\dot{y}}^2}) (\mu_{y\dot{y}} - \bar{\mu}_{y\dot{y}}) \}$, is not required for use in Eq. (6.30) because its "multiplier" was shown to be identically zero.

To obtain a general expression for $\mu_{jk}(t)$ defined by Eq. (6.32) that covers all three cases listed in Eq. (6.33), let us now define the generalized "instantaneous cross-correlation function" of the aircraft impulse response as

$$\phi_{h_j h_k}(\tau, t) \triangleq h_j(t - \frac{\tau}{2}) h_k(t + \frac{\tau}{2}), \quad (6.35)$$

where $h_j(t)$ and $h_k(t)$ are tabulated in Table 2 for the three cases listed in Eq. (6.33). Functions $\dot{h}(t)$ in Table 2 are the time-derivatives of the aircraft impulse response function $h(t)$. Since convolution of $\dot{h}(t)$ with the input yields $\dot{y}(t)$ as indicated by Eq. (1.21), it follows directly from Eqs. (K.14) and (K.15) in Appendix K that Table 2 contains the appropriate definitions of $h_j(t)$ and $h_k(t)$.

TABLE 1. VALUES OF THE PARAMETERS IN EQ. (6.34) THAT YIELD THE INDIVIDUAL EXPANSION COEFFICIENTS IN EQ. (6.30)

Coefficient	j'	k'	j''	k''
$E_{\sigma_f} \{ (\sigma_{\dot{y}}^2 - \overline{\sigma_{\dot{y}}^2})^2 \}$	2	0	2	0
$E_{\sigma_f} \{ (\sigma_{\ddot{y}}^2 - \overline{\sigma_{\ddot{y}}^2})^2 \}$	0	2	0	2
$E_{\sigma_f} \{ (\mu_{y\dot{y}} - \overline{\mu_{y\dot{y}}})^2 \}$	1	1	1	1
$E_{\sigma_f} \{ (\sigma_{\dot{y}}^2 - \overline{\sigma_{\dot{y}}^2}) (\sigma_{\ddot{y}}^2 - \overline{\sigma_{\ddot{y}}^2}) \}$	2	0	0	2
$E_{\sigma_f} \{ (\sigma_{\dot{y}}^2 - \overline{\sigma_{\dot{y}}^2}) (\mu_{y\dot{y}} - \overline{\mu_{y\dot{y}}}) \}$	2	0	1	1
$E_{\sigma_f} \{ (\sigma_{\ddot{y}}^2 - \overline{\sigma_{\ddot{y}}^2}) (\mu_{y\dot{y}} - \overline{\mu_{y\dot{y}}}) \}$	0	2	1	1

TABLE 2. DEFINITIONS OF $h_j(t)$ AND $h_k(t)$ FOR USE IN EQ. (6.35).

Response variable	j	k	$h_j(t)$	$h_k(t)$
$\mu_{jk} = \sigma_{\dot{y}}^2$	2	0	$h(t)$	$h(t)$
$\mu_{jk} = \sigma_{\ddot{y}}^2$	0	2	$\dot{h}(t)$	$\dot{h}(t)$
$\mu_{jk} = \mu_{y\dot{y}}$	1	1	$h(t)$	$\dot{h}(t)$

To apply the results of Appendix K to the computation of $\mu_{jk}(t)$, we further let

$$x_j(t) = x_k(t) = w_f(t) | \sigma_f(u) = \sigma_f(t) z(t) , \quad (6.36)$$

$$- \infty < u < \infty$$

in Appendix K, where $w_f(t)$, $\sigma_f(t)$, and $z(t)$ are components of our turbulence model of Eqs. (1.2) to (1.4), and where the conditioning operation in Eq. (6.36) indicates that $\sigma_f(t)$ is to be treated as a known function. With the interpretations provided by Eqs. (6.35) and (6.36), the instantaneous cross-spectrum input-response relation of Eq. (K.21) becomes

$$\Phi_{y_j y_k}(f, t | g_f) = \int_{-\infty}^{\infty} \Phi_{w_f}(f, t-u | g_f) \Phi_{h_j h_k}(f, u) du , \quad (6.37)$$

where $\Phi_{y_j y_k}(f, t | g_f)$ is the instantaneous cross-spectral density of $\mu_{jk}(t)$ [defined by Eq. (6.32)], $\Phi_{h_j h_k}(f, t)$ is the Fourier transform with respect to τ of Eq. (6.35) as interpreted by Table 2, and $\Phi_{w_f}(f, t | g_f)$ is the Fourier transform with respect to τ of

$$\phi_{w_f}(\tau, t | g_f) \triangleq \sigma_f(t - \frac{\tau}{2}) \sigma_f(t + \frac{\tau}{2}) \phi_z(\tau) , \quad (6.38)$$

where $\phi_z(\tau)$ is the autocorrelation function of the component $z(t)$ of the turbulence model of Eqs. (1.2) to (1.4), and $\sigma_f(t)$ is to be considered as a known function in Eq. (6.38).

When we apply the locally stationary approximation provided by the first term on the right-hand side of Eq. (2.45), we have

$$\Phi_{w_f}(f, t | g_f) \approx \sigma_f^2(t) \Phi_z(f) , \quad (6.39)$$

where $\Phi_z(f)$ is the Fourier transform of $\phi_z(\tau)$. By substituting Eq. (6.39) into Eq. (6.37) and integrating over all f , we obtain

$$\begin{aligned}
\mu_{jk}(t) &= \int_{-\infty}^{\infty} \Phi_{y_j y_k}(f, t | g_f) df \\
&= \int_{-\infty}^{\infty} \sigma_f^2(t-u) \int_{-\infty}^{\infty} \Phi_z(f) \Phi_{h_j h_k}(f, u) df du , \quad (6.40)
\end{aligned}$$

where we have interchanged orders of integration in going to the second line, and where we have used the fact that $\Phi_{y_j y_k}(f, t | g_f)$ is the conditional instantaneous cross-spectral density of $\{y_j^j(t)\}$ and $\{y_k^k(t)\}$, $j + k = 2$, whose integral over $-\infty < f < \infty$ yields $\mu_{jk}(t)$ — see Eq. (6.32) and the general property of Eq. (K.11). By extension of Eq. (2.54), let us now define

$$\gamma_{h_j h_k, z}(t) \triangleq \int_{-\infty}^{\infty} \Phi_z(f) \Phi_{h_j h_k}(f, t) df , \quad (6.41)$$

where as noted above, we have

$$\Phi_{h_j h_k}(f, t) \triangleq \int_{-\infty}^{\infty} h_j(t - \frac{\tau}{2}) h_k(t + \frac{\tau}{2}) e^{-i2\pi f \tau} d\tau, \quad (6.42)$$

where $h_j(t)$ and $h_k(t)$ have the interpretations listed in Table 2. Substituting Eq. (6.41) into Eq. (6.40) yields the desired form for $\mu_{jk}(t)$:

$$\mu_{jk}(t) = \int_{-\infty}^{\infty} \sigma_f^2(t-u) \gamma_{h_j h_k, z}(u) du . \quad (6.43)$$

In particular, taking the expected value of $\mu_{jk}(t)$ with respect to the process $\sigma_f(t)$ gives

$$\overline{\mu_{jk}} = \int_{-\infty}^{\infty} \overline{\sigma_f^2} \gamma_{h_j h_k, z}(u) du ; \quad (6.44)$$

hence, from Eqs. (6.43) and (6.44), we have

$$\mu_{jk}(t) - \overline{\mu_{jk}} = \int_{-\infty}^{\infty} [\sigma_f^2(t-u) - \overline{\sigma_f^2}] \gamma_{h_j h_k, z}(u) du . \quad (6.45)$$

We recall now that each coefficient in the left-hand column of Table 1 is of the form of Eq. (6.34); thus, each coefficient is the expected value with respect to $\{\sigma_f(t)\}$ of cross-products of the form of Eq. (6.45). Furthermore, we see that Eq. (6.45) is the convolution of the stochastic function $[\sigma_f^2(t) - \overline{\sigma_f^2}]$ with the "system characterization" $\gamma_{h_j h_k, z}(t)$. Thus, we can use the result, Eq. (K.26) of Appendix K, appropriately interpreted, to obtain a general formula for $E_{\sigma_f} \{(\mu_{j'k'} - \overline{\mu_{j'k'}})(\mu_{j''k''} - \overline{\mu_{j''k''}})\}$. To apply Eq. (K.26) to this situation, we require the Fourier transform of $\gamma_{h_j h_k, z}(t)$:

$$\tilde{\gamma}_{h_j h_k, z}(\nu) \triangleq \int_{-\infty}^{\infty} \gamma_{h_j h_k, z}(t) e^{i2\pi \nu t} dt . \quad (6.46)$$

Thus, applying Eq. (K.26) to the present situation gives

$$E_{\sigma_f} \{(\mu_{j'k'} - \overline{\mu_{j'k'}})(\mu_{j''k''} - \overline{\mu_{j''k''}})\} = \int_{-\infty}^{\infty} \Phi_{\sigma_f^2}(\nu) \tilde{\gamma}_{h_{j'}, h_{k'}, z}^*(-\nu) \tilde{\gamma}_{h_{j''}, h_{k''}, z}(-\nu) d\nu , \quad (6.47)$$

where $\Phi_{\sigma_f^2}(\nu)$ is the power spectral density of the process

$\{\sigma_f^2(t) - \overline{\sigma_f^2}\}$ which has no d-c value, the functions $\tilde{\gamma}_{h_j h_k}(v)$ are defined by Eq. (6.46) with primes added as appropriate, and the superscript asterisk denotes the complex conjugate. The negative arguments in Eq. (6.47) arise from the fact that we have used a positive exponent in Eq. (6.46), whereas the transforms $H_j(f)$ and $H_k(f)$ in Eq. (K.26) are defined as in Eq. (1.9) with negative exponents. Equation (6.47) is the main result of this subsection.

Expressions For System Characterizations

Here, we examine the system characterizations with respect to the process $\{z(t)\}$ that are defined by Eqs. (6.41) and (6.46). First, we note that $\gamma_{h_j h_k, z}(t)$ also can be expressed as

$$\gamma_{h_j h_k, z}(t) = \int_{-\infty}^{\infty} \Phi_z(f) \operatorname{Re} \Phi_{h_j h_k}(f, t) df \quad (6.48)$$

where $\operatorname{Re} \Phi_{h_j h_k}(f, t)$ denotes the real part of $\Phi_{h_j h_k}(f, t)$. To show this, we note from Eq. (6.42) that $\Phi_{h_j h_k}(f, t)$ can be expressed as

$$\begin{aligned} \Phi_{h_j h_k}(f, t) = & \int_{-\infty}^{\infty} h_j(t - \frac{\tau}{2}) h_k(t + \frac{\tau}{2}) \cos(2\pi f \tau) d\tau \\ & - i \int_{-\infty}^{\infty} h_j(t - \frac{\tau}{2}) h_k(t + \frac{\tau}{2}) \sin(2\pi f \tau) d\tau. \end{aligned} \quad (6.49)$$

Since $h_j(t)$ and $h_k(t)$ are real, the cosine integral in Eq. (6.49) is real, whereas i times the sine integral is imaginary. Moreover, the sine integral is an *odd* function of f . Since $\Phi_z(f)$ is an even function of f , the contribution of the sine integral in Eq. (6.49) to the integral $\gamma_{h_j h_k, z}(t)$

in Eq. (6.41) is zero. We are left with the contribution of the cosine integral, which is the real part of $\Phi_{h_j h_k}(f, t)$, as indicated by Eq. (6.48). Since $\Phi_z(f)$ is real, it follows that $\gamma_{h_j h_k, z}(t)$ also is a real function of t .

From the fact that $\gamma_{h_j h_k, z}(t)$ is real, it follows from Eq. (6.46) that

$$\tilde{\gamma}_{h_j h_k, z}(-v) = \tilde{\gamma}_{h_j h_k, z}^*(v) \quad (6.50)$$

where the asterisk denotes the complex conjugate. Hence, instead of Eq. (6.47), we also can write

$$E_{\sigma_f} \{ (\mu_{j, k}, -\bar{\mu}_{j, k}) (\mu_{j, k} - \bar{\mu}_{j, k}) \} = \int_{-\infty}^{\infty} \Phi_{\sigma_f^2}(v) \tilde{\gamma}_{h_j, h_k, z}(v) \tilde{\gamma}_{h_j, h_k, z}^*(v) dv. \quad (6.51)$$

Moreover, since both quantities $\mu_{j, k}(t) - \bar{\mu}_{j, k}$ in Eq. (6.51) are real as can be seen from Eq. (6.45) and the fact that $\gamma_{h_j h_k, z}(u)$ is real, by taking the complex conjugate of both sides of Eq. (6.51) it follows that we also have the second alternative form

$$E_{\sigma_f} \{ (\mu_{j, k}, -\bar{\mu}_{j, k}) (\mu_{j, k} - \bar{\mu}_{j, k}) \} = \int_{-\infty}^{\infty} \Phi_{\sigma_f^2}(v) \tilde{\gamma}_{h_j, h_k, z}^*(v) \tilde{\gamma}_{h_j, h_k, z}(v) dv, \quad (6.52)$$

where we have used the fact that the power spectral density $\Phi_{\sigma_f^2}(v)$ of $\{\sigma_f^2(t) - \sigma_f^2\}$ is real.

We see - e.g., from Eq. (6.51) - that the transformed system characterization $\tilde{y}_{h_j h_k, z}(v)$ defined by Eq. (6.46)

is the quantity required for computation of the expansion coefficients $E_{\sigma_f} \{(\mu_{j,k}, -\bar{\mu}_{j,k})(\mu_{j,k}, -\bar{\mu}_{j,k})\}$. Inserting

Eq. (6.41) into Eq. (6.46) and interchanging orders of integration yields

$$\tilde{y}_{h_j h_k, z}(v) = \int_{-\infty}^{\infty} \Phi_z(f) \int_{-\infty}^{\infty} \Phi_{h_j h_k}(f, t) e^{i2\pi vt} dt df \quad (6.53)$$

$$= \int_{-\infty}^{\infty} \Phi_z(f) H_j^*(f + \frac{v}{2}) H_k(f - \frac{v}{2}) df \quad (6.54)$$

where we have used Eq. (K.23), and Eq. (K.8) applied to $h_j h_k$. Functions $H_j(f)$ and $H_k(f)$ in Eq. (6.54) are the Fourier transforms of $h_j(t)$ and $h_k(t)$ as defined by Eq. (1.9) where individual interpretations of $h_j(t)$ and $h_k(t)$ are given in Table 2.

According to Table 2, we require three forms of $\tilde{y}_{h_j h_k, z}(v)$ which are $\tilde{y}_{hh, z}(v)$, $\tilde{y}_{\dot{h}\dot{h}, z}(v)$, and $\tilde{y}_{h\dot{h}, z}(v)$.

We can immediately write out the first of these three forms using Eq. (6.54) as

$$\tilde{y}_{hh, z}(v) = \int_{-\infty}^{\infty} \Phi_z(f) H(f - \frac{v}{2}) H^*(f + \frac{v}{2}) df, \quad (6.55)$$

which we have discussed earlier - see Eq. (2.62).

In order to put $\tilde{y}_{\dot{h}\dot{h}, z}(v)$ and $\tilde{y}_{h\dot{h}, z}(v)$ in the form of Eq. (6.54), we require the Fourier transform of the time derivative of $h(t)$. Let us denote this Fourier transform by placing a dot over $H(f)$ - i.e., we define $\dot{H}(f)$ as

$$\dot{H}(f) \triangleq \int_{-\infty}^{\infty} \dot{h}(t) e^{-i2\pi ft} dt \quad (6.56)$$

The Fourier mate to Eq. (6.56) is

$$\dot{h}(t) = \int_{-\infty}^{\infty} \dot{H}(f) e^{i2\pi ft} df . \quad (6.57)$$

Furthermore, the Fourier mate to Eq. (1.9) is

$$h(t) = \int_{-\infty}^{\infty} H(f) e^{i2\pi ft} df . \quad (6.58)$$

Differentiating both sides of Eq. (6.58) with respect to t gives

$$\dot{h}(t) = \int_{-\infty}^{\infty} i2\pi f H(f) e^{i2\pi ft} df ; \quad (6.59)$$

hence, comparing Eqs. (6.57) and (6.59) yields

$$\dot{H}(f) = i2\pi f H(f) , \quad (6.60)$$

which relates the Fourier transform $\dot{H}(f)$ of $\dot{h}(t)$ to the Fourier transform $H(f)$ of $h(t)$.

Using Eqs. (6.54) and (6.60), we now can express $\tilde{Y}_{\dot{h}\dot{h},z}(\nu)$ as

$$\begin{aligned} \tilde{Y}_{\dot{h}\dot{h},z}(\nu) &= \int_{-\infty}^{\infty} \Phi_z(f) (-i)2\pi(f + \frac{\nu}{2}) H^*(f + \frac{\nu}{2}) (i)2\pi(f - \frac{\nu}{2}) H(f - \frac{\nu}{2}) df \\ &= 4\pi^2 \int_{-\infty}^{\infty} f^2 \Phi_z(f) H(f - \frac{\nu}{2}) H^*(f + \frac{\nu}{2}) df \\ &\quad - \pi^2 \nu^2 \int_{-\infty}^{\infty} \Phi_z(f) H(f - \frac{\nu}{2}) H^*(f + \frac{\nu}{2}) df . \end{aligned} \quad (6.61)$$

From Eq. (6.55), we see that the integral in the second term is $\tilde{\gamma}_{hh,z}(\nu)$; hence, $\tilde{\gamma}_{\dot{h}\dot{h},z}(\nu)$ can be expressed as

$$\tilde{\gamma}_{\dot{h}\dot{h},z}(\nu) = 4\pi^2 \int_{-\infty}^{\infty} f^2 \Phi_z(f) H(f - \frac{\nu}{2}) H^*(f + \frac{\nu}{2}) df - \pi^2 \nu^2 \tilde{\gamma}_{hh,z}(\nu) . \quad (6.62)$$

However, the autocorrelation function of the process $\{\dot{z}(t)\}$ whose sample functions are the time derivatives of $z(t)$ is the negative of the second derivative of the autocorrelation function of $\{z(t)\}$ — i.e., $-\phi_z''(\tau)$ [see, for example, p. 21 of Ref. 37]. Furthermore, by differentiating twice with respect to τ the relationship

$$\phi_z(\tau) = \int_{-\infty}^{\infty} \Phi_z(f) e^{i2\pi f\tau} df \quad (6.63)$$

one finds

$$\phi_z''(\tau) = - \int_{-\infty}^{\infty} 4\pi^2 f^2 \Phi_z(f) e^{i2\pi f\tau} df , \quad (6.64)$$

from which it follows that $4\pi^2 f^2 \Phi_z(f)$ is the power spectral density of the process $\{\dot{z}(t)\}$. We therefore can express $\tilde{\gamma}_{\dot{h}\dot{h},z}(\nu)$ as

$$\tilde{\gamma}_{\dot{h}\dot{h},z}(\nu) = \tilde{\gamma}_{hh,\dot{z}}(\nu) - \pi^2 \nu^2 \tilde{\gamma}_{hh,z}(\nu) \quad (6.65)$$

where we have defined

$$\tilde{\gamma}_{hh,\dot{z}}(\nu) \triangleq 4\pi^2 \int_{-\infty}^{\infty} f^2 \Phi_z(f) H(f - \frac{\nu}{2}) H^*(f + \frac{\nu}{2}) df . \quad (6.66)$$

Finally, we consider $\tilde{\gamma}_{\dot{h}\dot{h},z}(\nu)$. By combining Eqs. (6.54) and (6.60), we have in this case

$$\begin{aligned}
\tilde{\gamma}_{hh,z}(\nu) &= \int_{-\infty}^{\infty} \Phi_z(f) H^*(f + \frac{\nu}{2}) (i) 2\pi (f - \frac{\nu}{2}) H(f - \frac{\nu}{2}) df \\
&= i2\pi \int_{-\infty}^{\infty} f \Phi_z(f) H(f - \frac{\nu}{2}) H^*(f + \frac{\nu}{2}) df \\
&\quad - i\pi\nu \int_{-\infty}^{\infty} \Phi_z(f) H(f - \frac{\nu}{2}) H^*(f + \frac{\nu}{2}) df .
\end{aligned} \tag{6.67}$$

However, from Eqs. (35a) and (35b) of Ref. 34, we see that $[H(f - \frac{\nu}{2}) H^*(f + \frac{\nu}{2})]$ is necessarily an even function of f since $\Phi_h(f, t)$ is an even function of f . Furthermore, $\Phi_z(f)$ also is an even function of f . Hence, we always have

$$\int_{-\infty}^{\infty} f \Phi_z(f) H(f - \frac{\nu}{2}) H^*(f + \frac{\nu}{2}) df = 0 \tag{6.68}$$

since the integrand in Eq. (6.68) is an odd function of f . Thus, combining Eq. (6.67) with Eqs. (6.55) and (6.68), we have

$$\tilde{\gamma}_{hh,z}(\nu) = -i\pi\nu \tilde{\gamma}_{hh,z}(\nu) , \tag{6.69}$$

which is the desired expression for $\tilde{\gamma}_{hh,z}(\nu)$.

Summary. Equations (6.55), (6.65)-(6.66), and (6.69) provide expressions for $\tilde{\gamma}_{hh,z}(\nu)$, $\tilde{\gamma}_{\dot{h}\dot{h},z}(\nu)$, and $\tilde{\gamma}_{h\dot{h},z}(\nu)$ as a function of the power spectral density $\Phi_z(f)$ of the process $\{z(t)\}$ of the turbulence model of Eqs. (1.2) through (1.4) and the complex frequency response function $H(f)$, Eq. (1.9), of the relevant aircraft response variable. These expressions for $\tilde{\gamma}_{hh,z}(\nu)$, $\tilde{\gamma}_{\dot{h}\dot{h},z}(\nu)$, and $\tilde{\gamma}_{h\dot{h},z}(\nu)$ are to be combined with Eq. (6.51) to obtain the various coefficients $E_{\sigma_f}\{(\mu_{j'k'} - \bar{\mu}_{j'k'}) (\mu_{j''k''} - \bar{\mu}_{j''k''})\}$ by applying the rules in

Tables 1 and 2. By carrying out this procedure, we shall now show that the second to the last coefficient in the left-hand column of Table 1 is identically zero.

According to Tables 1 and 2, we have from Eq. (6.51) for the second to the last coefficient in the left-hand column of Table 1:

$$\begin{aligned} E_{\sigma_f} \{(\sigma_y^2 - \overline{\sigma_y^2})(\mu_{y\dot{y}} - \overline{\mu_{y\dot{y}}})\} &= \int_{-\infty}^{\infty} \Phi_{\sigma_f^2}(\nu) \tilde{\gamma}_{hh,z}(\nu) \tilde{\gamma}_{hh,z}^*(\nu) d\nu \\ &= i\pi \int_{-\infty}^{\infty} \nu \Phi_{\sigma_f^2}(\nu) |\tilde{\gamma}_{hh,z}(\nu)|^2 d\nu, \quad (6.70) \end{aligned}$$

where we have used Eq. (6.69) in going to the second line. The integral in the second line of the right-hand side of Eq. (6.70) is real; hence, the entire right-hand side is imaginary. However, the left-hand side of Eq. (6.70) is real. It follows that the integral in the second line of the right-hand side must be zero. This property also follows from the fact that the integrand in the second line of Eq. (6.70) is an odd function of ν since $\Phi_{\sigma_f^2}(\nu)$ and $|\tilde{\gamma}_{hh,z}(\nu)|^2$ are both even functions of ν . The even property of $|\tilde{\gamma}_{hh,z}(\nu)|^2$ follows from Wiener's theorem and the fact that $\gamma_{hh,z}(t)$ is real. Equation (135) of Ref. 34 is a statement of Wiener's theorem, where we remind the reader that $\rho_w(\tau)$ is an even function of τ . We therefore have

$$E_{\sigma_f} \{(\sigma_y^2 - \overline{\sigma_y^2})(\mu_{y\dot{y}} - \overline{\mu_{y\dot{y}}})\} \equiv 0. \quad (6.71)$$

The validity of Eq. (6.71) also could be argued on physical grounds from the fact that it is the coefficient in Eq. (6.30) of an *odd* function of the response variable y .

Furthermore, we shall now show that

$$\overline{\mu_{y\dot{y}}} \triangleq E_{\sigma_f} \{\mu_{y\dot{y}}(t)\} = 0 \quad (6.72)$$

as was stated earlier without proof by Eq. (6.13). From Eqs. (6.44) and (6.46), we have

$$\bar{\mu}_{jk} = \overline{\sigma_f^2} \int_{-\infty}^{\infty} \gamma_{h_j h_k, z}(t) dt = \overline{\sigma_f^2} \tilde{\gamma}_{h_j h_k, z}(0) . \quad (6.73)$$

Hence, for $j = 1$ and $k = 1$, we have according to Table 2 and Eq. (6.73),

$$\bar{\mu}_{y\dot{y}} = \overline{\sigma_f^2} \tilde{\gamma}_{h\dot{h}, z}(0) = 0 , \quad (6.74)$$

where the right-hand equality follows from Eq. (6.69).

Final Expression For Aircraft Response Exceedance Rates

Incorporating the result, Eq. (6.71) into Eq. (6.30), we obtain now our final expression for the mean rate of exceedances $N_+(y)$ of our aircraft response variable past the level y :

$$\begin{aligned} N_+(y) = & \frac{1}{2\pi} \left(\frac{\overline{\sigma_y^2}}{\overline{\sigma_{\dot{y}}^2}} \right)^{\frac{1}{2}} e^{-\frac{y^2}{2\overline{\sigma_y^2}}} \left\{ 1 - \frac{E_{\sigma_f} \{ (\sigma_y^2 - \overline{\sigma_y^2})^2 \}}{8(\overline{\sigma_y^2})^2} \right. \\ & + \frac{E_{\sigma_f} \{ (\sigma_y^2 - \overline{\sigma_y^2})^2 \}}{8(\overline{\sigma_y^2})^2} \left[3 - 6 \frac{y^2}{\overline{\sigma_y^2}} + \frac{y^4}{(\overline{\sigma_y^2})^2} \right] \\ & - \frac{1}{4} \left(\frac{2E_{\sigma_f} \{ (\mu_{y\dot{y}} - \bar{\mu}_{y\dot{y}})^2 \} + E_{\sigma_f} \{ (\sigma_y^2 - \overline{\sigma_y^2})(\sigma_{\dot{y}}^2 - \overline{\sigma_{\dot{y}}^2}) \}}{\overline{\sigma_y^2} \overline{\sigma_{\dot{y}}^2}} \right) \left(1 - \frac{y^2}{\overline{\sigma_y^2}} \right) \Bigg\} \\ & + \text{higher order terms}, \end{aligned} \quad (6.75)$$

where the "higher order terms" are the same as those in Eq. (6.30), which arise from the higher order terms in Eq. (6.10). Neglect of these higher-order terms should not lead to substantial error provided that *typical* fluctuations in the modulating process $\sigma_f(t)$ are not larger than about one-third of the mean value of $\sigma_f(t)$. The turbulence model for which the result Eq. (6.75) applies is that of Eqs. (1.2) to (1.4), where here the "slow" component $w_s(t)$ is taken to be negligible in comparison with the "fast" component $w_f(t) = \sigma_f(t)z(t)$.

To evaluate the parameters and coefficients in Eq. (6.75), we require

$$\overline{\sigma_y^2} = \int_{-\infty}^{\infty} \Phi_{w_f}(f) |H(f)|^2 df \quad (6.76)$$

and

$$\overline{\sigma_{\dot{y}}^2} = \int_{-\infty}^{\infty} \Phi_{w_f}(f) |\dot{H}(f)|^2 df \quad (6.77a)$$

$$= 4\pi^2 \int_{-\infty}^{\infty} f^2 \Phi_{w_f}(f) |H(f)|^2 df, \quad (6.77b)$$

where $\Phi_{w_f}(f)$ is the power spectral density of the turbulence process $\{w_f(t)\}$, $H(f)$ is the complex frequency response of the aircraft response variable of interest as defined by Eq. (1.9), and $\dot{H}(f)$ is the Fourier transform of the time derivative of the impulse response function of the aircraft response variable of interest. Equation (6.77a) is a direct consequence of Eq. (1.21), and Eq. (6.77b) follows from Eq. (6.60). Furthermore, from Eq. (6.51) and Tables 1 and 2, we find that the coefficients in Eq. (6.75) can be expressed as

$$E_{\sigma_f} \{ (\sigma_y^2 - \overline{\sigma_y^2})^2 \} = \int_{-\infty}^{\infty} \Phi_{\sigma_f^2}(\nu) |\tilde{\gamma}_{hh,z}(\nu)|^2 d\nu \quad (6.78)$$

$$E_{\sigma_f} \{ (\sigma_{\dot{y}}^2 - \overline{\sigma_{\dot{y}}^2})^2 \} = \int_{-\infty}^{\infty} \Phi_{\sigma_f^2}(\nu) |\tilde{\gamma}_{\dot{h}\dot{h},z}(\nu)|^2 d\nu \quad (6.79)$$

$$E_{\sigma_f} \{ (\mu_{y\dot{y}} - \overline{\mu_{y\dot{y}}})^2 \} = \int_{-\infty}^{\infty} \Phi_{\sigma_f^2}(\nu) |\tilde{\gamma}_{h\dot{h},z}(\nu)|^2 d\nu \quad (6.80)$$

and

$$\begin{aligned} E_{\sigma_f} \{ (\sigma_y^2 - \overline{\sigma_y^2})(\sigma_{\dot{y}}^2 - \overline{\sigma_{\dot{y}}^2}) \} \\ = \int_{-\infty}^{\infty} \Phi_{\sigma_f^2}(\nu) \tilde{\gamma}_{hh,z}(\nu) \tilde{\gamma}_{\dot{h}\dot{h},z}^*(\nu) d\nu . \end{aligned} \quad (6.81)$$

$\Phi_{\sigma_f^2}(\nu)$ is the power spectral density of the process $\{\sigma_f^2(t) - \overline{\sigma_f^2}\}$. Two methods for computing this spectrum are given on pp. 79-83 of Ref. 19. System characterizations (with respect to the process $\{z(t)\}$) $\tilde{\gamma}_{hh,z}(\nu)$, $\tilde{\gamma}_{\dot{h}\dot{h},z}(\nu)$, and $\tilde{\gamma}_{h\dot{h}}(\nu)$ are given by Eqs. (6.55), (6.65)-(6.66), and (6.69) respectively. $\Phi_z(f)$ is the power spectral density of the process $\{z(t)\}$ which can be taken as the normalized spectrum $\Phi_{w_f}(f) / \int_{-\infty}^{\infty} \Phi_{w_f}(f) df$ because of the locally stationary assumption Eq. (1.8a), and the normalization assumption, Eq. (1.4).

Limiting cases of the result given by Eq. (6.75) can be studied in a manner analogous to that used in Secs. 2.5 and 2.6. It can be shown that the result given by Eq. (6.75) reduces to the appropriate result in Sec. 4.3 of Ref. 19 when fluctuations of $\sigma_f(t)$ are assumed to be negligible over the duration of the aircraft impulse response $h(t)$.

APPENDIX A

DERIVATION OF CORRECTION TERMS TO THE GAUSSIAN PROBABILITY DENSITY FUNCTION FOR USE IN EQUATION (2.12)

Equation (2.9) expresses the correction terms in Eq. (2.12) as derivatives of the Gaussian density function:

$$p^{(k)}(y|\sigma_f) = \frac{d^k}{d(\sigma_y^2)^k} \frac{1}{\sqrt{2\pi\sigma_y^2}} e^{-\frac{y^2}{2\sigma_y^2}} \bigg|_{\sigma_y^2 = \sigma_f^2}, \quad (A.1)$$

where

$$p^{(0)}(y|\sigma_f) = \frac{1}{\sqrt{2\pi\sigma_y^2}} e^{-\frac{y^2}{2\sigma_y^2}}. \quad (A.2)$$

Differentiating Eq. (A.1) with respect to σ_y^2 yields

$$\begin{aligned} p^{(1)}(y|\sigma_f) &= \frac{1}{\sqrt{2\pi}} \left[\left(-\frac{1}{2}\right) (\sigma_y^2)^{-3/2} e^{-\frac{y^2}{2\sigma_y^2}} \right. \\ &\quad \left. + (\sigma_y^2)^{-1/2} e^{-\frac{y^2}{2\sigma_y^2}} \frac{y^2}{2} (\sigma_y^2)^{-2} \right] \\ &= \frac{p(y|\sigma_f)}{2\sigma_y^2} \left[\frac{y^2}{\sigma_y^2} - 1 \right], \end{aligned} \quad (A.3)$$

in agreement with Eq. (2.14a). In similar manner, we find

$$\begin{aligned} p^{(2)}(y|\sigma_f) &= \frac{d}{d(\sigma_y^2)} p^{(1)}(y|\sigma_f) \\ &= \frac{p(y|\sigma_f)}{4(\sigma_y^2)^2} \left[\frac{y^4}{(\sigma_y^2)^2} - 6 \frac{y^2}{\sigma_y^2} + 3 \right], \end{aligned} \quad (A.4)$$

and

$$\begin{aligned}
 p^{(3)}(y|\tilde{\sigma}_f) &= \frac{d}{d(\sigma_y^2)} p^{(2)}(y|\tilde{\sigma}_f) \\
 &= \frac{p(y|\sigma_f)}{8(\sigma_y^2)^3} \left[\frac{y^6}{(\sigma_y^2)^3} - 15 \frac{y^4}{(\sigma_y^2)^2} \right. \\
 &\quad \left. + 45 \frac{y^2}{\sigma_y^2} - 15 \right] , \tag{A.5}
 \end{aligned}$$

in agreement with Eqs. (2.15b) and (2.16b).

APPENDIX B

DERIVATION OF EQUATION (2.23)

Equation (2.23) is derived by substituting Eq. (2.20) into Eq. (2.19c) and performing the resulting integration:

$$p(y) = \frac{1}{\sqrt{2\pi}} \int_0^\infty \frac{1}{\sqrt{\sigma_y^2}} e^{-\frac{y^2}{2\sigma_y^2}} \frac{\gamma}{\sigma_y^2 \Gamma(\gamma)} (\gamma \sigma_y^2 / \overline{\sigma_y^2})^{\gamma-1} \times e^{-\gamma \sigma_y^2 / \overline{\sigma_y^2}} d\sigma_y^2 . \quad (B.1)$$

Let us define

$$\xi \triangleq \frac{\gamma}{\sigma_y^2} \sigma_y^2 ; \quad (B.2)$$

hence,

$$d\xi = \frac{\gamma}{\sigma_y^2} d\sigma_y^2 . \quad (B.3)$$

Substituting Eqs. (B.2) and (B.3) into Eq. (B.1) yields

$$p(y) = \frac{1}{\sqrt{2\pi(\sigma_y^2/\gamma)} \Gamma(\gamma)} \int_0^\infty e^{-\frac{y^2}{2(\sigma_y^2/\gamma)\xi}} \times \xi^{\gamma-3/2} e^{-\xi} d\xi . \quad (B.4)$$

Let us now define the normalized variable

$$\eta = \frac{y}{\sqrt{\sigma_y^2}} . \quad (B.5)$$

Since $d\eta = dy/\sqrt{\sigma_y^2}$, the probability density of η is

$$\begin{aligned}
 p(\eta) &= \frac{1}{\sqrt{2\pi/\gamma} \Gamma(\gamma)} \int_0^\infty e^{-\frac{\gamma\eta^2}{2\xi}} \xi^{\gamma-3/2} e^{-\xi} d\xi \\
 &= \frac{1}{\sqrt{2\pi/\gamma} \Gamma(\gamma)} 2 \left(\frac{\gamma^{1/2} |\eta|}{\sqrt{2}} \right)^{\gamma-1/2} K_{\gamma-1/2} (\sqrt{2\gamma} |\eta|) \\
 &= \frac{\sqrt{2\gamma/\pi}}{2^{\gamma-1/2} \Gamma(\gamma)} (\sqrt{2\gamma} |\eta|)^{\gamma-1/2} K_{\gamma-1/2} (\sqrt{2\gamma} |\eta|), \tag{B.6}
 \end{aligned}$$

where Eq. (3.471.9) on p. 340 of Gradshteyn and Ryzhik [50] was used in going to the second line, and where $K_n(\cdot)$ is the modified Bessel function of the second kind of order n . Equation (B.6) is the same as Eq. (2.23) in the main text.

APPENDIX C

CLOSED FORM EXPRESSIONS FOR $K_n(x)$ AND $p_\gamma(\eta)$ IN TERMS OF ELEMENTARY FUNCTIONS

From Eq. (4) on p. 108 of Relton [51], we have the recursion relation

$$K_{n+1}(x) = \frac{2n}{x} K_n(x) + K_{n-1}(x) \quad , \quad (C.1)$$

whereas, from Eq. (8) on p. 109 of Ref. 47, we have

$$K_{\frac{1}{2}}(x) = K_{-\frac{1}{2}}(x) = \left(\frac{\pi}{2x}\right)^{\frac{1}{2}} e^{-x} \quad . \quad (C.2)$$

Combining Eqs. (C.1) and (C.2) gives

$$K_{\frac{3}{2}}(x) = \left(1 + \frac{1}{x}\right) K_{\frac{1}{2}}(x) \quad . \quad (C.3)$$

Combining Eqs. (C.2) and (C.3) with Eq. (C.1) gives

$$K_{\frac{5}{2}}(x) = \left(1 + \frac{3}{x} + \frac{3}{x^2}\right) K_{\frac{1}{2}}(x) \quad . \quad (C.4)$$

In like manner, combining each previous two values of $K_n(x)$ with Eq. (C.1) yields, successively:

$$K_{\frac{7}{2}}(x) = \left(1 + \frac{6}{x} + \frac{15}{x^2} + \frac{15}{x^3}\right) K_{\frac{1}{2}}(x) \quad (C.5)$$

$$K_{\frac{9}{2}}(x) = \left(1 + \frac{10}{x} + \frac{45}{x^2} + \frac{105}{x^3} + \frac{105}{x^4}\right) K_{\frac{1}{2}}(x) \quad (C.6)$$

$$K_{\frac{11}{2}}(x) = \left(1 + \frac{15}{x} + \frac{105}{x^2} + \frac{420}{x^3} + \frac{945}{x^4} + \frac{945}{x^5}\right) K_{\frac{1}{2}}(x) \quad (C.7)$$

$$K_{\frac{13}{2}}(x) = \left(1 + \frac{21}{x} + \frac{210}{x^2} + \frac{1260}{x^3} + \frac{4725}{x^4} + \frac{10395}{x^5} + \frac{10395}{x^6}\right) \times K_{\frac{1}{2}}(x) \quad (C.8)$$

$$K_{15/2}(x) = \left(1 + \frac{28}{x} + \frac{378}{x^2} + \frac{3150}{x^3} + \frac{17325}{x^4} + \frac{62370}{x^5} + \frac{135135}{x^6} + \frac{135135}{x^7} \right) K_{\frac{1}{2}}(x) \quad (C.9)$$

$$K_{17/2}(x) = \left(1 + \frac{36}{x} + \frac{630}{x^2} + \frac{6930}{x^3} + \frac{51975}{x^4} + \frac{270270}{x^5} + \frac{945945}{x^6} + \frac{2027025}{x^7} + \frac{2027025}{x^8} \right) K_{\frac{1}{2}}(x) \quad (C.10)$$

$$K_{19/2}(x) = \left(1 + \frac{45}{x} + \frac{990}{x^2} + \frac{13860}{x^3} + \frac{135135}{x^4} + \frac{945945}{x^5} + \frac{4729725}{x^6} + \frac{16216200}{x^7} + \frac{34459425}{x^8} + \frac{34459425}{x^9} \right) K_{\frac{1}{2}}(x), \quad (C.11)$$

where $K_{\frac{1}{2}}(x)$ is given in terms of elementary functions by Eq. (C.2).

Combining Eqs. (C.2), (C.3), (C.5), (C.9), and (C.10), successively, with Eq. (2.23), we obtain the following expressions for $p_Y(\eta)$ defined by Eq. (2.23):

$$p_1(\eta) = \frac{1}{\sqrt{2}} e^{-\sqrt{2}|\eta|} \quad (C.12)$$

$$p_2(\eta) = \frac{1}{2} (2|\eta|+1) e^{-2|\eta|} \quad (C.13)$$

$$p_4(\eta) = \frac{1}{2^{7/2} \times 3} (x^3 + 6x^2 + 15x + 15) e^{-x} ,$$

$$\text{where } x = \sqrt{8}|\eta| \quad (C.14)$$

$$p_8(\eta) = \frac{1}{322560} (x^7 + 28x^6 + 378x^5 + 3150x^4 \\ + 17325x^3 + 62370x^2 + 135135x + 135135) e^{-x} ,$$

$$\text{where } x = \sqrt{16}|\eta| \quad (C.15)$$

$$p_9(\eta) = \frac{3}{2^{17/2} 8!} (x^8 + 36x^7 + 630x^6 + 6930x^5 \\ + 51975x^4 + 270270x^3 + 945945x^2 \\ + 2027025x + 2027025) e^{-x} ,$$

$$\text{where } x = \sqrt{18}|\eta| \text{ and } 8! = 40320. \quad (C.16)$$

APPENDIX D

ALTERNATIVE SERIES REPRESENTATION OF EXPANSION COEFFICIENT OF THE NONGAUSSIAN TERM

Writing v for f in Eq. (2.58), we may express the coefficient $\mu_{\sigma_y^2}^{(2)}$ in Eq. (2.24) as

$$\mu_{\sigma_y^2}^{(2)} = \int_{-\infty}^{\infty} \phi_{\sigma_f^2}(v) |\tilde{\gamma}_{h,z}(v)|^2 dv . \quad (D.1)$$

Let us define

$$\phi_{\gamma}(t) \triangleq \int_{-\infty}^{\infty} \gamma_{h,z}(\xi) \gamma_{h,z}(\xi+t) dt , \quad (D.2)$$

which is the "autocorrelation function" of the deterministic system characterization $\gamma_{h,z}(t)$ defined by Eq. (2.54). Then, from Wiener's theorem — e.g., p. 54 of Ref. 34 — it follows that

$$\phi_{\gamma}(t) = \int_{-\infty}^{\infty} |\tilde{\gamma}_{h,z}(v)|^2 e^{i2\pi vt} dv . \quad (D.3)$$

Furthermore, from Eqs. (2.81), and (D.3), it follows by applying the generalized form of Parseval's theorem to Eq. (D.1) that we may express $\mu_{\sigma_y^2}^{(2)}$ as

$$\begin{aligned} \mu_{\sigma_y^2}^{(2)} &= \int_{-\infty}^{\infty} \phi_{\sigma_f^2}(t) \phi_{\gamma}(t) dt \\ &= 2 \int_0^{\infty} \phi_{\sigma_f^2}(t) \phi_{\gamma}(t) dt , \end{aligned} \quad (D.4)$$

where the second line follows from the fact that $\phi_{\sigma_f^2}(t)$ and $\phi_Y(t)$ both are necessarily even functions of t .

Let us now consider situations where $\sigma_f(t)$ fluctuates little over durations comparable with the duration of the aircraft impulse response $h(t)$. In these situations, the nominal correlation time of $\sigma_f^2(t)$ is large in comparison with that of $\gamma_{h,z}(t)$ — as is illustrated in Fig. D.1. Examination of Fig. D.1 suggests that we represent $\phi_{\sigma_f^2}(t)$ by a low-order polynomial over the range $0 \leq t \leq T_Y$, where T_Y is the time interval over which $\phi_Y(t)$ is not negligible for $t > 0$:

$$\phi_{\sigma_f^2}(t) = \sum_{j=0}^n b_j t^j, \quad 0 \leq t \leq T_Y, \quad (D.5)$$

where the b_j may be interpreted as the one-sided Maclaurin expansion coefficients of $\phi_{\sigma_f^2}(t)$,

$$b_j = \frac{\phi_{\sigma_f^2}^{(j)}(0+)}{j!}, \quad (D.6)$$

where $\phi_{\sigma_f^2}^{(j)}(0+)$ denotes the j th "right-hand" derivative of $\phi_{\sigma_f^2}(t)$ evaluated at the origin. Substitution of Eq. (D.5) into Eq. (D.4) yields the desired series representation of $\mu_{\sigma_y^2}^{(2)}$:

$$\mu_{\sigma_y^2}^{(2)} = 2 \sum_{j=0}^n b_j \int_0^\infty t^j \phi_Y(t) dt. \quad (D.7)$$

Using Eq. (D.6), we see that the first term in the expansion, Eq. (D.7), is

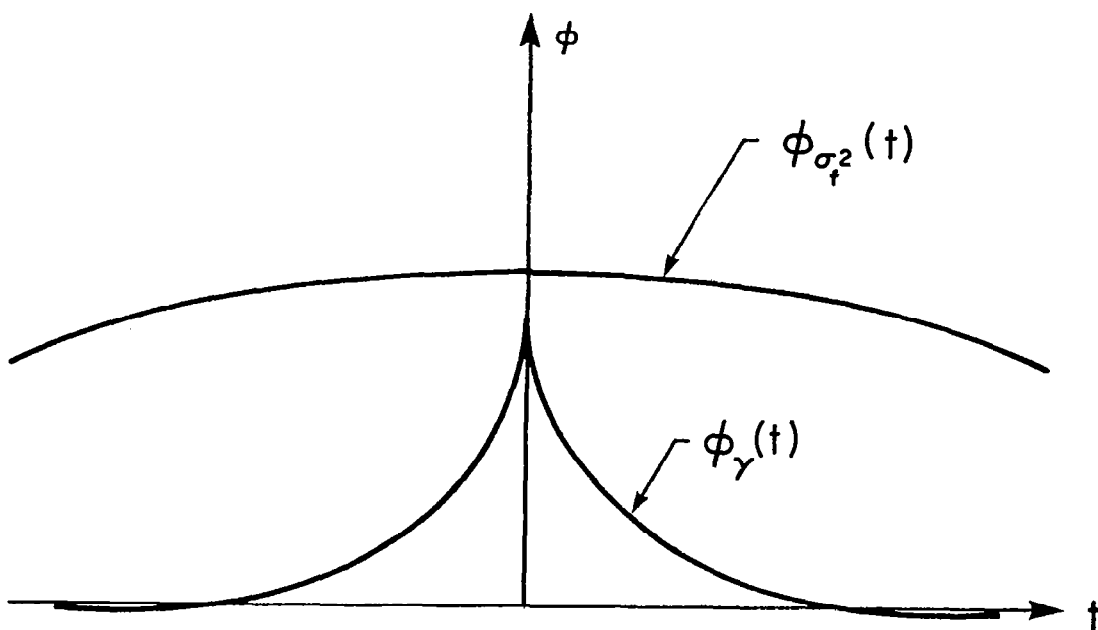


FIG. D.1. BEHAVIOR OF $\phi_{\sigma_f^2}(t)$ AND $\phi_\gamma(t)$ NEAR $t = 0$ FOR CASE WHERE VARIATION IN $\sigma_f^2(t)$ IS SMALL OVER DURATION OF $h(t)$.

$$\begin{aligned}
2 b_0 \int_0^{\infty} \phi_Y(t) dt &= \phi_{\sigma_f^2}(0) \int_{-\infty}^{\infty} \phi_Y(t) dt \\
&= \phi_{\sigma_f^2}(0) \left[\int_{-\infty}^{\infty} \gamma_{h,z}(t) dt \right]^2 \\
&= \phi_{\sigma_f^2}^{(0)} \left(\Gamma_{h,z}^{(0)} \right)^2, \tag{D.8}
\end{aligned}$$

where the second line was obtained using Eq. (D.2), and the third line was obtained using Eqs. (2.87) and (2.92). Comparing Eqs. (2.96) and (D.8), we see that the first terms in the series of Eqs. (2.96) and (D.7) are the same. However, the remaining terms differ because of the presence of odd powers of j in Eq. (D.7).

Although the first correction term to the term represented by Eq. (D.8) in the series of Eq. (2.85) had a very satisfying interpretation as we showed in Eq. (2.99), the series, Eq. (D.7), generally will be better behaved in practice because it utilizes the "one-sided" expansion of $\phi_{\sigma_f^2}(t)$ shown in Eqs. (D.5) and (D.6) which does *not* require that $\dot{\phi}_{\sigma_f^2}(t)$ be continuous at the origin. Thus, the existence

of $E \left\{ \left(\frac{d\sigma_f^2}{dt} \right)^2 \right\}$ is not required by the expansion of Eq. (D.7), whereas, it is required in the expansion of Eq. (2.85) — see Eq. (2.93).

Finally, we note that the "one-sided moments" of $\phi_Y(t)$ in Eq. (D.7) can be evaluated from the derivatives of the unilateral Laplace transform of $\phi_Y(t)$ if that transform can be evaluated in closed form. Also notice that the characterization of $\phi_{\sigma_f^2}(t)$ used in Eq. (D.5) is its one-sided power series expansion indicated at the very end of Sec. 1.

APPENDIX E

PROOF THAT REAL AND IMAGINARY PARTS OF FOURIER SERIES COEFFICIENTS OF A PERIODIC RANDOM PROCESS ARE UNCORRELATED FOR $m \neq n$, WHERE $m, n \geq 0$

In Eq. (3.5), we have expressed the complex Fourier series coefficients of a periodic random process

$$w(t+pT) = w(t) , \quad p=0, \pm 1, \pm 2, \dots \quad (\text{E.1})$$

by

$$c_n = \frac{1}{T} \int_{-T/2}^{T/2} w(t) e^{-i2\pi nt/T} dt \quad (\text{E.2a})$$

$$= a_n - ib_n , \quad (\text{E.2b})$$

where a_n and b_n are real. Consider

$$\begin{aligned} c_n c_m^* &= (a_n - ib_n) (a_m + ib_m) \\ &= (a_n a_m + b_n b_m) + i(a_n b_m - a_m b_n) \end{aligned} \quad (\text{E.3})$$

and

$$c_n c_m = (a_n a_m - b_n b_m) - i(a_n b_m + a_m b_n) . \quad (\text{E.4})$$

Taking the expected values of Eqs. (E.3) and (E.4), we find that if

$$E\{c_n c_m^*\} = 0 \quad \text{and} \quad E\{c_n c_m\} = 0 , \quad (\text{E.5})$$

then we must have

$$E\{a_n a_m\} = -E\{b_n b_m\} = E\{b_n b_m\} \equiv 0 \quad (\text{E.6})$$

and

$$E\{a_n b_m\} = E\{a_m b_n\} = -E\{a_m b_n\} \equiv 0 \quad . \quad (E.7)$$

Satisfaction of the *two* conditions in Eq. (E.5) therefore guarantees that all pairs of the real and imaginary parts of the complex coefficients c_n are uncorrelated. Cf. Davenport and Root [39], pp. 91,92.

From Eq. (E.2a), we have

$$\begin{aligned} E\{c_n c_m^*\} &= \frac{1}{T^2} \int_{-T/2}^{T/2} \int_{-T/2}^{T/2} E\{w(t_1) w(t_2)\} e^{-i2\pi(nt_1 - mt_2)/T} dt_1 dt_2 \\ &= \frac{1}{T^2} \int_{-T/2}^{T/2} \int_{-T/2}^{T/2} \phi_w(t_2 - t_1) e^{-i2\pi(nt_1 - mt_2)/T} dt_1 dt_2 \quad . \end{aligned} \quad (E.8)$$

If, in the inner integral, we transform t_2 to τ using

$$\tau = t_2 - t_1 ; \quad (E.9)$$

hence, $t_2 = \tau + t_1$, we have

$$e^{-i2\pi(nt_1 - mt_2)/T} = e^{-i2\pi(n-m)t_1/T} e^{i2\pi m\tau/T} ; \quad (E.10)$$

therefore,

$$\begin{aligned} E\{c_n c_m^*\} &= \frac{1}{T^2} \int_{-T/2}^{T/2} e^{-i2\pi(n-m)t_1/T} \int_{-T/2-t_1}^{T/2-t_1} \phi_w(\tau) e^{i2\pi m\tau/T} d\tau dt_1 \\ &= \frac{1}{T} \int_{-T/2}^{T/2} e^{-i2\pi(n-m)t_1/T} dt_1 \\ &\quad \times \frac{1}{T} \int_{-T/2}^{T/2} \phi_w(\tau) e^{i2\pi m\tau/T} d\tau \quad , \end{aligned} \quad (E.11)$$

where the second line is a consequence of the fact that the autocorrelation function $\phi_w(\tau)$ of a periodic random process is itself periodic with the same period T — i.e., Davenport and Root [39], p. 91 — and the fact that the inner integral in the first line in Eq. (E.11) is independent of t_1 , since its integrand is periodic with period T . Furthermore, for $m \neq n$, the first integral in Eq. (E.11) is identically zero; hence,

$$E\{c_n c_m^*\} = 0 \quad , \quad m \neq n \quad , \quad (E.12)$$

which is the first of our two conditions in Eq. (E.5).

To check the second condition, we note from Eq. (E.2a) that $c_m = c_{-m}^*$, and hence, from Eq. (E.11), we have

$$\begin{aligned} E\{c_n c_m\} &= E\{c_n c_{-m}^*\} \\ &= \frac{1}{T} \int_{-T/2}^{T/2} e^{-i2\pi(n+m)t_1/T} dt_1 \\ &\quad \times \frac{1}{T} \int_{-T/2}^{T/2} \phi_w(\tau) e^{-i2\pi m\tau/T} d\tau \quad . \end{aligned} \quad (E.13)$$

Consequently,

$$E\{c_n c_m\} = 0 \quad , \quad m \neq -n \quad . \quad (E.14)$$

It follows that Eqs. (E.6) and (E.7) are satisfied for all $m, n \geq 0$, provided $m \neq n$, which is what we sought to prove.

As pointed out in the main text, when $w(t)$ is generated from a stationary Gaussian process, the entire set of variates a_n, b_m is jointly Gaussian since Eq. (E.2a) is a linear transformation of $w(t)$. Hence, provided $m \neq n$, for all $m, n > 0$, all a_n 's and b_m 's are statistically independent.

APPENDIX F

METHOD FOR SOLUTION OF EQUATION (3.26) FOR INTEGRAL SCALE OF von KARMAN TURBULENCE

The likelihood equation for the integral scale of turbulence with negligible low frequency component $w_s(t)$ is given by Eq. (3.26):

$$\sum_{i=1}^N \left[\frac{d}{dL} \ln F_i(L) \right] \left[\frac{S_i}{F_i(L)} - \frac{1}{N} \sum_{j=1}^N \frac{S_j}{F_j(L)} \right] = 0, \quad (F.1)$$

where we have reversed the roles of i and j in comparison with Eq. (3.26). Following the notation of Eq. (3.30), we define

$$G_i(L) \triangleq \frac{d}{dL} \ln F_i(L). \quad (F.2)$$

Let us now substitute Eq. (F.2) into Eq. (F.1) and define after minor rearrangement

$$E(L) \triangleq \frac{1}{N} \sum_{i=1}^N G_i(L) \left\{ \left[\frac{1}{N} \sum_{j=1}^N \frac{S_j}{F_j(L)} \right] - \frac{S_i}{F_i(L)} \right\} \quad (F.3)$$

Then, according to Eq. (F.3), the value of L that satisfies Eq. (F.1) is the value for which $E(L) \equiv 0$.

To illustrate a method for obtaining the solution $E(L) = 0$ of Eq. (F.3), we consider a vertical turbulence velocity record that is assumed to obey the von Karman (transverse) power spectral form. From Eqs. (3.34) and (3.35) we have for the von Karman transverse spectrum:

$$F_i(L) \equiv \frac{1 + 188.75 L^2 k_i^2}{[1 + 70.78 L^2 k_i^2]^{11/6}} \quad (F.4)$$

and

$$G_i(L) \equiv \frac{1}{L} \frac{117.97L^2k_i^2(1-188.75L^2k_i^2)}{(1+70.78L^2k_i^2)(1+188.75L^2k_i^2)} \quad . \quad (F.5)$$

The solution $E(L) = 0$ of Eq. (F.3) is most easily obtained by trial and error. In carrying out the solution for the vertical record under consideration, we used a value for the uppermost wavenumber corresponding to $j = N$ in Eq. (F.3) of $k_N = 3.0 \times 10^{-2}$ cycles/meter which yielded a value of $N = 6326$ points in the summation in Eq. (F.3).

The actual value of $L = 309.4$ m was obtained as follows. First, a trial value of $L = 305$ m (1000 ft) was chosen. The value of $E(L)$ for this trial value of L was then computed and stored using Eq. (F.3). Since the value of $E(L)$ obtained was negative, a second (larger) trial value of L was chosen which was 335.5 meters (1100 ft). Using Eq. (F.3), a new value of $E(L)$ was computed and stored using this second choice of L . We then possessed two values of $E(L)$ corresponding to the two trial values of L . Linear interpolation then was used to estimate a new (3rd) value of L corresponding to the value of $E(L) = 0$. For this third value of L , the true value of $E(L)$ was then computed using Eq. (F.3). The resulting value of $E(L)$ had a positive sign; hence, a new (fourth) trial value of L was chosen which was 3.05 m (10 ft) smaller than the third trial value of L . The value of $E(L)$ corresponding to this fourth trial value of L was then computed and stored using Eq. (F.3). Finally, from the third and fourth trial values of L and the corresponding values of $E(L)$, a fifth value of L was computed by linear interpolation corresponding to the value $E(L) = 0$. This fifth value of $L = 309.4$ m was used as the solution to the likelihood equation (F.1). These five values of L and the corresponding values of $E(L)$ are plotted in Fig. F.1, which shows the local very nearly linear behavior of $E(L)$ as a function of L .

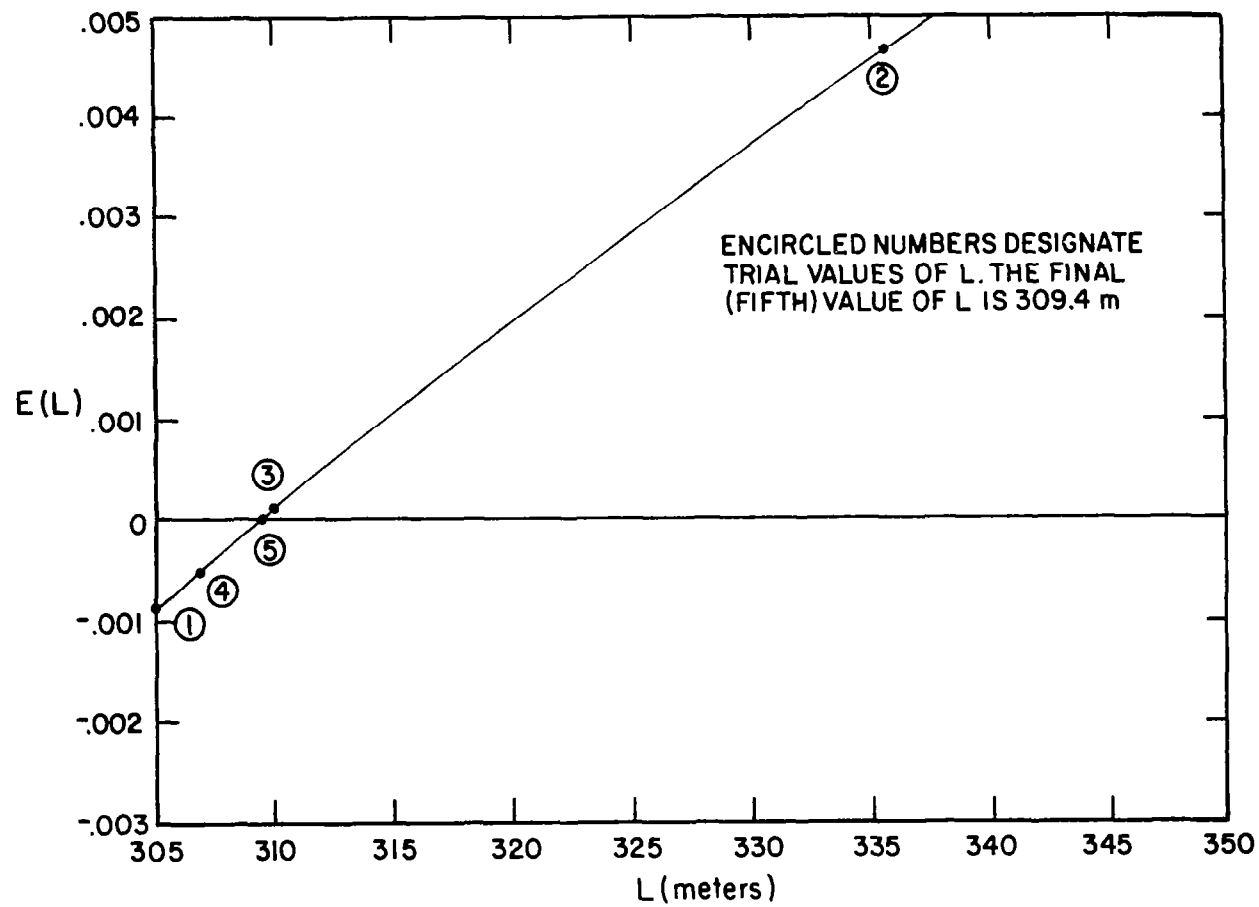


FIG. F.1. TRIAL AND ERROR SOLUTION FOR INTEGRAL SCALE L.

APPENDIX G

TRADE-OFFS BETWEEN CHOICES OF ξ_H AND m

Here, we describe considerations to be taken into account in choosing values of ξ_H and m in the constrained least-squares estimation procedure described in Sec. 4. Upper limit ξ_H in Eq. (4.5) determines the interval $0 \leq \xi \leq \xi_H$ over which the parameters σ_f^2 , L , and a_0 to a_m in the autocorrelation function model of Eq. (4.1) are obtained by minimization of the integral squared error E in the constrained least-squares fit procedure. The parameter m is the degree of the polynomial in Eq. (4.1) that is used to represent the autocorrelation function of the low-frequency turbulence component $w_s(t)$ over the interval $0 \leq \xi \leq \xi_H$.

Intelligent choices for values of ξ_H and m to be used in the minimization procedure are not generally independent. One reason for this lack of independence is the fact that our representation in Eqs. (4.1) and (4.5) of the von Karman component $\sigma_f^2 \phi_K(\xi; L)$ of the autocorrelation function is not orthogonal with our representation of the low-frequency component $\sum_{i=0}^m a_i \xi^i$ of the autocorrelation function over the interval $0 \leq \xi \leq \xi_H$. Thus, the low-frequency component autocorrelation function representation $\sum_{i=0}^m a_i \xi^i$ has the potential for representing a portion of the *von Karman component* of the empirical autocorrelation function $R(\xi)$ in Eq. (4.5) in the integral squares sense. However, if for given values of ξ_H and m , $\sum_{i=0}^m a_i \xi^i$ can represent, exactly, the *low-frequency component* $R(\xi)$ over $0 \leq \xi \leq \xi_H$, and if the "fast" component of $R(\xi)$ has exactly the appropriate von Karman form for some values of L and σ_f^2 , then this lack of orthogonality between $\sigma_f^2 \phi_K(\xi; L)$ and $\sum_{i=0}^m a_i \xi^i$ will not be a problem. Thus, our goal should be to choose values of ξ_H and m so that $\sum_{i=0}^m a_i \xi^i$ can do a good job of representing the low-frequency component of the empirical autocorrelation function $R(\xi)$, while simultaneously attempting to minimize the capability of $\sum_{i=0}^m a_i \xi^i$ to represent the *von Karman component* of $R(\xi)$.

To aid in implementing the above italicized rule, let us now consider the capability of $\sum_{i=0}^m a_i \xi^i$ to represent the von Karman component $\sigma^2 \phi_K(\xi; L)$ of the autocorrelation function over the interval $0 \leq \xi \leq \xi_H$. Thus, for both the von Karman transverse and longitudinal autocorrelation functions $\phi_K(\xi; L)$, we determine the set of coefficients a_i , $i = 0, 1, \dots, m$ that minimize the integral-square "error"

$$E \triangleq \int_0^{\xi_H} [\sigma^2 \phi_K(\xi; L) - \sum_{i=0}^m a_i \xi^i]^2 d\xi. \quad (G.1)$$

To find the a_i that minimize E , we differentiate Eq. (G.1) with respect to a_j , $j = 0, 1, \dots, m$ which yields

$$\frac{\partial E}{\partial a_j} = -2 \int_0^{\xi_H} [\sigma^2 \phi_K(\xi; L) - \sum_{i=0}^m a_i \xi^i] \xi^j d\xi. \quad (G.2)$$

The solution to the set of equations $(\partial E / \partial a_j) = 0$, $j = 0, 1, \dots, m$ determines the set of a_j that minimize E . From Eq. (G.2), this set of equations can be written as

$$\sigma^2 \int_0^{\xi_H} \xi^j \phi_K(\xi; L) d\xi - \sum_{i=0}^m a_i \int_0^{\xi_H} \xi^{i+j} d\xi = 0, \quad j = 0, 1, \dots, m \quad (G.3)$$

or

$$\sum_{i=0}^m \frac{\xi_H^{i+j+1}}{i+j+1} a_i = \sigma^2 \int_0^{\xi_H} \xi^j \phi_K(\xi; L) d\xi, \quad j = 0, 1, \dots, m, \quad (G.4)$$

which is a set of $m+1$ linear simultaneous algebraic equations for the a_i , $i = 0, 1, \dots, m$ in terms of ξ_H and von Karman autocorrelation function $\sigma^2 \phi_K(\xi; L)$. These a_i minimize E .

To write these equations in normalized form, let us define, as before,

$$\bar{\xi} \triangleq \frac{\xi}{L}, \quad \bar{\xi}_H \triangleq \frac{\xi_H}{L}, \quad (G.5a,b)$$

and

$$\phi_K(\xi;L) \equiv \bar{\phi}_K(\xi/L) = \bar{\phi}_K(\bar{\xi}), \quad (G.6)$$

as in Eqs. (4.18) and (4.20). Using Eqs. (G.5) and (G.6), we have, after dividing Eq. (G.4) by $\sigma^2 L^{j+1}$,

$$\sum_{i=0}^m \frac{\bar{\xi}_H^{i+j+1}}{i+j+1} \frac{L^i a_i}{\sigma^2} = \int_0^{\bar{\xi}_H} \bar{\xi}^j \bar{\phi}_K(\bar{\xi}) d\bar{\xi}, \quad j = 0, 1, \dots, m, \quad (G.7)$$

which is a set of equations for the normalized coefficients $L^i a_i / \sigma^2$. For a given value of m , the normalized solutions to these equations depend on only one dimensionless parameter $\bar{\xi}_H$. Thus, for either the von Karman transverse or longitudinal autocorrelation functions, we can solve the set of equations (G.7) for $L^i a_i / \sigma^2$, $i = 0, 1, \dots, m$ for any family of choices of $\bar{\xi}_H = \xi_H / L$ and m .

We shall find it convenient to have an explicit formula for the least integral-square "error" E in terms of the normalized solution vector $L^i a_i / \sigma^2$, $i = 0, 1, \dots, m$ to the set of equations (G.7). Squaring the integrand in Eq. (G.1) and rearranging terms, we have

$$E = \sigma^4 \int_0^{\xi_H} \phi_K^2(\xi;L) d\xi - 2\sigma^2 \sum_{i=0}^m a_i \int_0^{\xi_H} \xi^i \phi_K(\xi;L) d\xi + \int_0^{\xi_H} \left(\sum_{i=0}^m a_i \xi^i \right)^2 d\xi, \quad (G.8)$$

or dividing by $\sigma^4 L$ and introducing the notation of Eq. (G.6), we have

$$\begin{aligned}
\frac{E}{\sigma^2 L} &= \int_0^{\xi_H} \overline{\phi}_K^2(\xi/L) d\xi/L - 2 \sum_{i=0}^m \frac{L^i a_i}{\sigma^2} \int_0^{\xi_H} \frac{\xi^i}{L^i} \overline{\phi}_K(\xi/L) d\xi/L \\
&+ \int_0^{\xi_H} \left(\sum_{i=0}^m \frac{L^i a_i}{\sigma^2} \frac{\xi^i}{L^i} \right)^2 d\xi/L \\
&= \int_0^{\overline{\xi}_H} \overline{\phi}_K^2(\overline{\xi}) d\overline{\xi} - 2 \sum_{i=0}^m \frac{L^i a_i}{\sigma^2} \int_0^{\overline{\xi}_H} \overline{\xi}^i \overline{\phi}_K(\overline{\xi}) d\overline{\xi} \\
&+ \int_0^{\overline{\xi}_H} \left(\sum_{i=0}^m \frac{L^i a_i}{\sigma^2} \overline{\xi}^i \right)^2 d\overline{\xi} , \tag{G.9}
\end{aligned}$$

where we have introduced the notation of Eq. (G.5a) and (G.5b). However, expanding the last term in Eq. (G.9) yields

$$\begin{aligned}
\int_0^{\overline{\xi}_H} \left(\sum_{i=0}^m \frac{L^i a_i}{\sigma^2} \overline{\xi}^i \right)^2 d\overline{\xi} &= \sum_{i=0}^m \sum_{j=0}^m \frac{L^i a_i}{\sigma^2} \frac{L^j a_j}{\sigma^2} \int_0^{\overline{\xi}_H} \overline{\xi}^{i+j} d\overline{\xi} \\
&= \sum_{i=0}^m \sum_{j=0}^m \frac{L^i a_i}{\sigma^2} \frac{L^j a_j}{\sigma^2} \frac{\overline{\xi}_H^{i+j+1}}{i+j+1} \\
&= \sum_{j=0}^m \frac{L^j a_j}{\sigma^2} \sum_{i=0}^m \frac{L^i a_i}{\sigma^2} \frac{\overline{\xi}_H^{i+j+1}}{i+j+1} \\
&= \sum_{j=0}^m \frac{L^j a_j}{\sigma^2} \int_0^{\overline{\xi}_H} \overline{\xi}^j \overline{\phi}_K(\overline{\xi}) d\overline{\xi} \\
&= \sum_{i=0}^m \frac{L^i a_i}{\sigma^2} \int_0^{\overline{\xi}_H} \overline{\xi}^i \overline{\phi}_K(\overline{\xi}) d\overline{\xi} , \tag{G.10}
\end{aligned}$$

where the second to the last line follows from Eq. (G.7); therefore the last two lines are valid only for the dimensionless solution vector $L^i a_i / \sigma^2$, $i = 0, 1, \dots, m$ that minimizes E . Substitution of Eq. (G.10) into Eq. (G.9) yields

$$\frac{E}{\sigma^4 L} = \int_0^{\xi_H} \bar{\phi}_K^2(\bar{\xi}) d\bar{\xi} - \sum_{i=0}^m \frac{L^i a_i}{\sigma^2} \int_0^{\xi_H} \bar{\xi}^i \bar{\phi}_K(\bar{\xi}) d\bar{\xi}, \quad (G.11)$$

which is valid only for the set of dimensionless coefficients $L^i a_i / \sigma^2$, $i = 0, 1, \dots, m$ that is the solution to Eq. (G.7) — i.e., the set that minimizes E .

If all $a_i \equiv 0$, we have from Eq. (G.1),

$$\frac{E}{\sigma^4 L} = \int_0^{\xi_H} \phi_K^2(\xi; L) d\xi / L \equiv \int_0^{\xi_H} \bar{\phi}_K^2(\bar{\xi}) d\bar{\xi}, \quad (G.12)$$

where the right-hand equality follows from Eqs. (G.5) and (G.6). Hence, let us define a normalized "error" \bar{E} as

$$\bar{E} \triangleq \frac{E / \sigma^4 L}{\int_0^{\xi_H} \bar{\phi}_K^2(\bar{\xi}) d\bar{\xi}}. \quad (G.13)$$

Furthermore, let us define a normalized set of coefficients by

$$\bar{a}_i \triangleq \frac{L^i a_i}{\sigma^2}. \quad (G.14)$$

Using this latter definition, our set of equations (G.7) for the a_i 's can be written as

$$\sum_{i=0}^m \frac{\bar{\xi}^{i+j+1}}{i+j+1} \bar{a}_i = \int_0^{\xi_H} \bar{\xi}^j \bar{\phi}_K(\bar{\xi}) d\bar{\xi}, \quad j = 0, 1, \dots, m, \quad (G.15)$$

whereas dividing Eq. (G.11) by $\int_0^{\bar{\xi}_H} \bar{\phi}_K^2(\bar{\xi}) d\bar{\xi}$ and using the definitions of Eqs. (G.13) and (G.14), our equation for the normalized "error" \bar{E} becomes

$$\bar{E} = 1 - \frac{\sum_{i=0}^m \bar{a}_i \int_0^{\bar{\xi}_H} \bar{\xi}^i \bar{\phi}_K(\bar{\xi}) d\bar{\xi}}{\int_0^{\bar{\xi}_H} \bar{\phi}_K^2(\bar{\xi}) d\bar{\xi}} . \quad (G.16)$$

For either the von Karman transverse or longitudinal autocorrelation functions $\bar{\phi}_K(\bar{\xi})$, Eqs. (G.15) and (G.16) determine the normalized "error" \bar{E} in the least-squares best

fit of $\sum_{i=0}^m \bar{a}_i \bar{\xi}^i$ to the von Karman autocorrelation function

$\bar{\phi}_K(\bar{\xi})$. This value of \bar{E} is dependent on choices of $\bar{\xi}_H$ and m . Figures G.1 and G.3 show this dependence of \bar{E} on $\bar{\xi}_H$ and m for the von Karman transverse autocorrelation function, and Figs. G.2 and G.4 show the same dependence for the von Karman longitudinal autocorrelation function. *According to our above italicized statement, large values of \bar{E} are desirable. Therefore, for several representations*

$$\tilde{\phi}(\xi) = \overline{\sigma_f^2} \phi_K(\xi; L) + \sum_{i=0}^m \bar{a}_i \xi^i , \quad 0 \leq \xi \leq \xi_H \quad (G.17)$$

of an empirical autocorrelation function $R(\xi)$, all having approximately the same capability for representing the low-frequency component of $R(\xi)$, but differing in values of ξ_H and m , the representation with the pair of values $\xi_H = L\xi_H$ and m yielding the largest value of \bar{E} as determined by Figs. G.1 to G.4 should yield the most reliable value of L .

This rule of thumb suggests the following procedure for estimating autocorrelation parameters by the methodology of Sec. 4.

1. Compute the empirical autocorrelation function $R(\xi)$ — e.g., by the method outlined in Reference 18.
2. By visual inspection, choose the *largest* value of ξ_H for which $\sum_{i=0}^m a_i \xi^i$ can be expected to provide a good representation of the low-frequency component of $R(\xi)$ for each of several values of m , say $m = 1, 2, 3$, and 4. A different value of ξ_H will generally be chosen for each different value of m .
3. For each pair of values of ξ_H and m , compute L , σ_f^2 , a_0, a_1, \dots, a_m by the method described in Sec. 4. Plot the resulting representation, Eq. G.17, against $R(\xi)$ to insure adequacy of the fit for each such computation.
4. For each such fit — i.e., for each pair of values of ξ_H and m — determine the value of \bar{E} from Figs. G.1 or G.3, or from Figs. G.2 or G.4, as appropriate for the von Karman transverse or longitudinal cases.
5. The most reliable fit, i.e., the most reliable value of L , should be that corresponding to the *largest* value of \bar{E} . Values of $\xi_H = L\bar{\xi}_H$ and m yielding values of \bar{E} less than, say, 0.5 may be particularly unreliable because in this range $\sum_{i=0}^m a_i \xi^i$ has too much capability for representing a portion of the von Karman component of the empirical autocorrelation function $R(\xi)$.

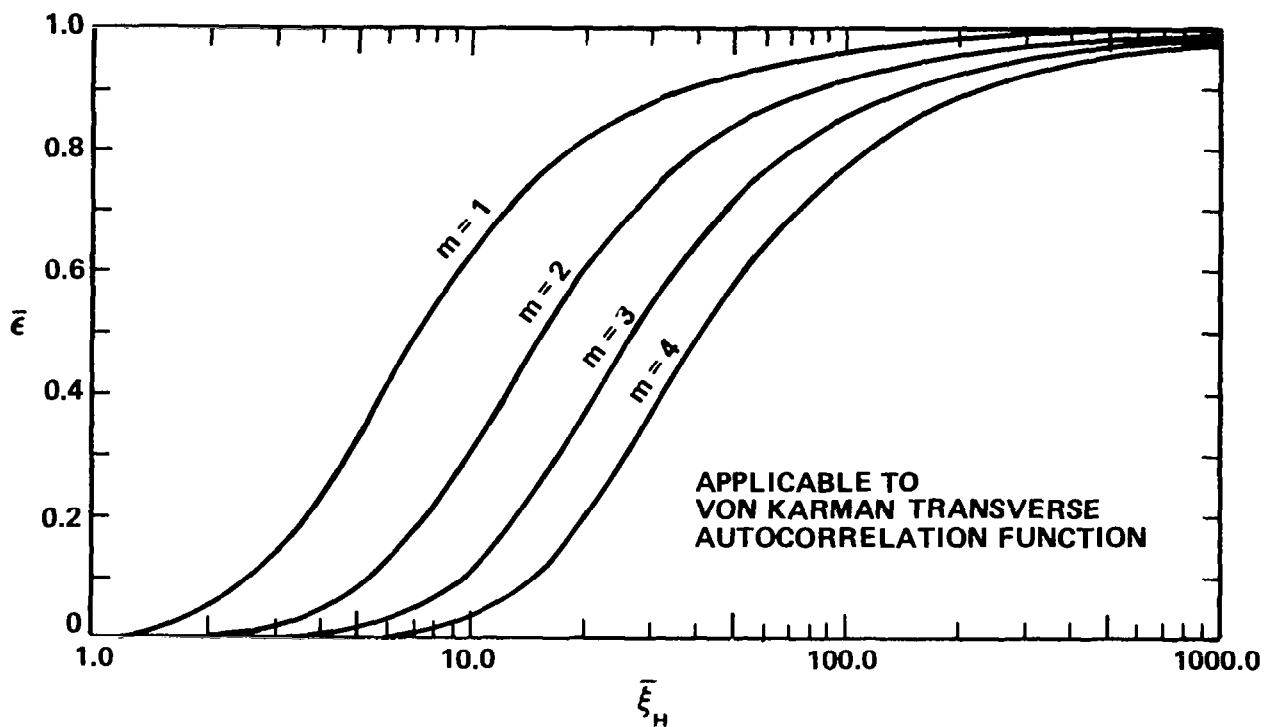


FIG. G.1. NORMALIZED "ERROR" EQ. (G.16) IN LEAST-SQUARES
FIT OF $\sum_{i=0}^m a_i \xi^i$ TO von KARMAN TRANSVERSE AUTO-
CORRELATION FUNCTION.

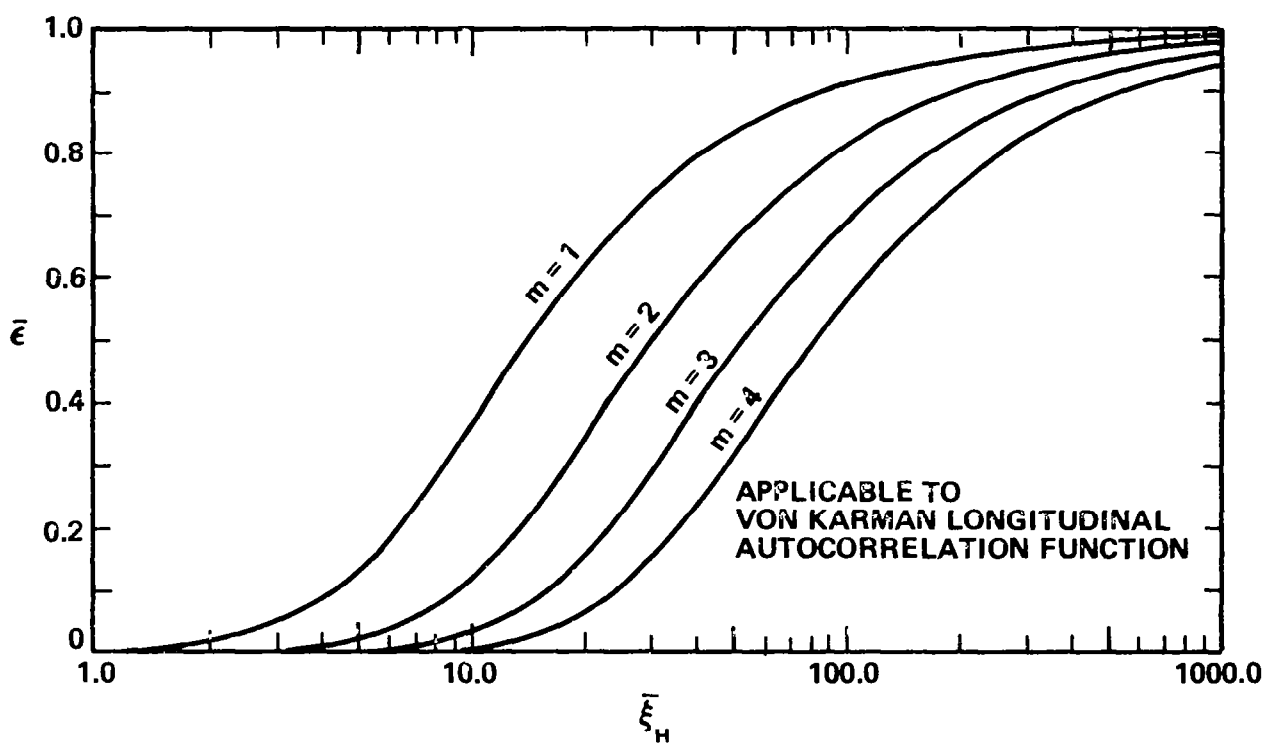


FIG. G.2. NORMALIZED "ERROR" EQ. (G.16) IN LEAST-SQUARES
 FIT OF $\sum_{i=0}^m a_i \xi^i$ TO von KARMAN LONGITUDINAL
 AUTOCORRELATION FUNCTION.

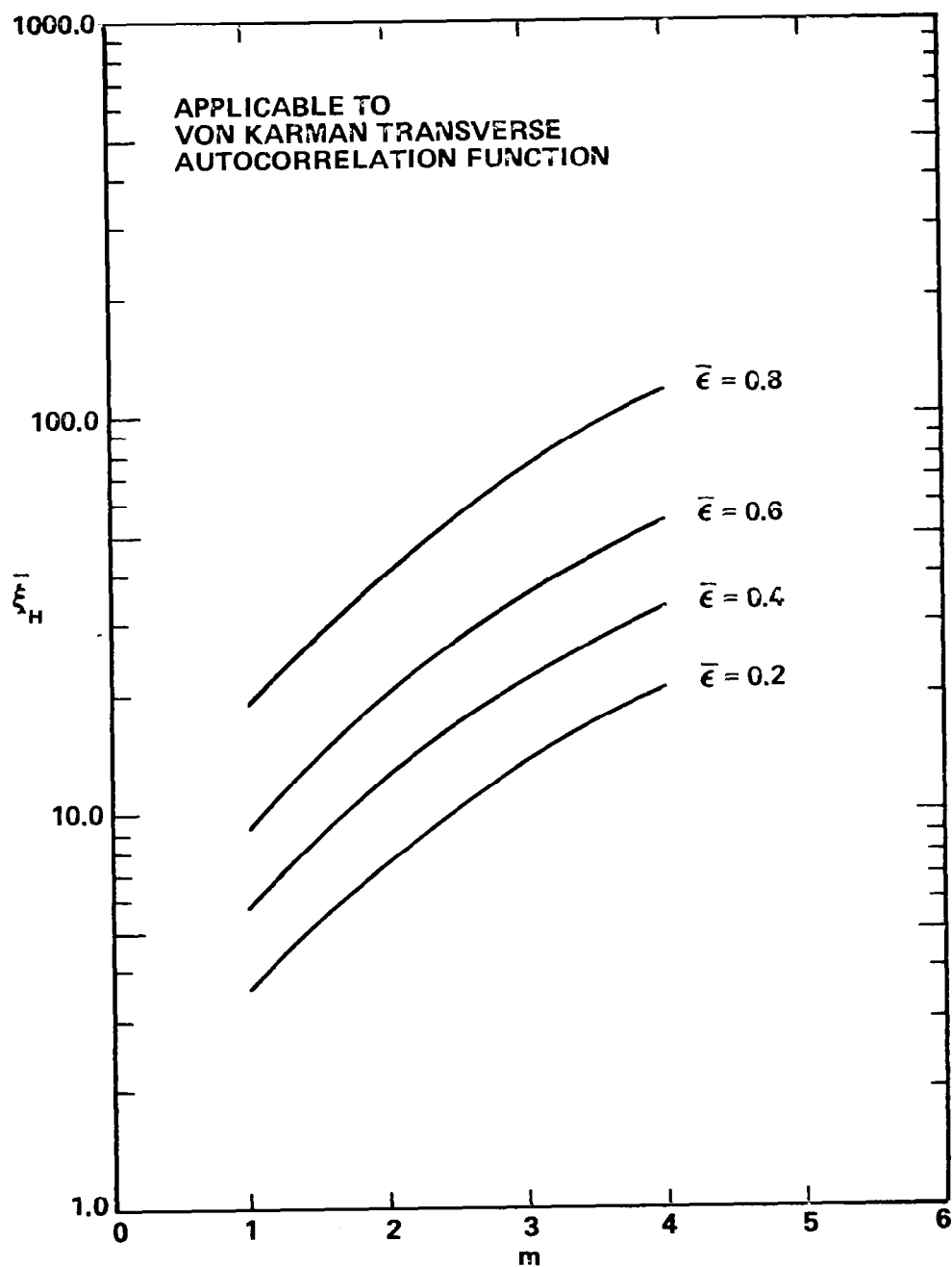


FIG. G.3. NORMALIZED "ERROR" EQ. (G.16) IN LEAST-SQUARES
FIT OF $\sum_{i=0}^m a_i \xi^i$ TO von KARMAN TRANSVERSE AUTO-
CORRELATION FUNCTION.

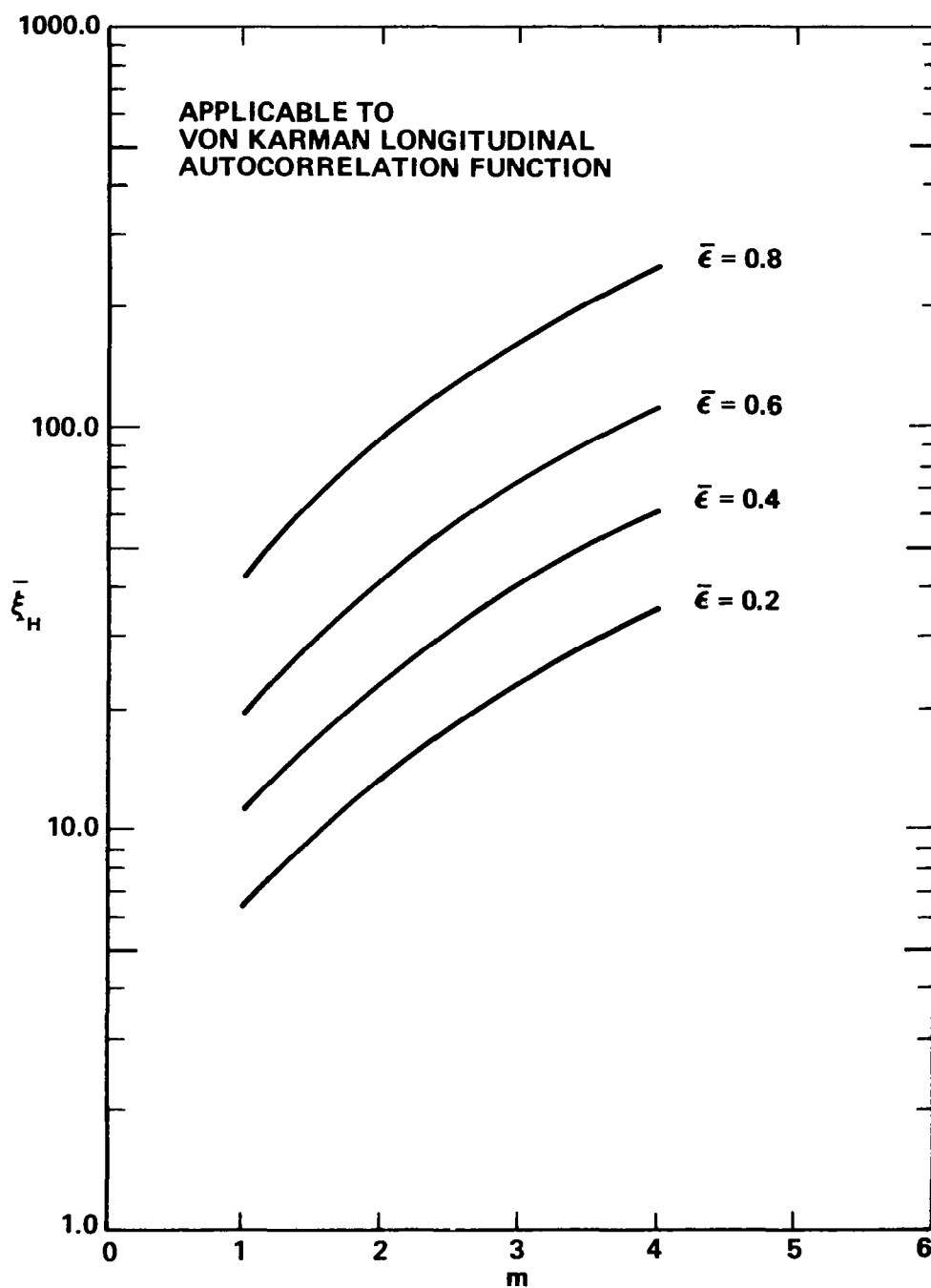


FIG. G.4. NORMALIZED "ERROR" EQ. (G.16) IN LEAST-SQUARES
FIT OF $\sum_{i=0}^m a_i \xi^i$ TO von KARMAN LONGITUDINAL
AUTOCORRELATION FUNCTION.

APPENDIX H

METHOD OF APPROXIMATION OF THE INTEGRAL SCALE AND POWER SPECTRUM OF THE "SLOW" TURBULENCE COMPONENT $w_s(t)$

Extrapolation of autocorrelation function model. Here, we develop a simple method of extrapolating the autocorrelation function approximation $\phi_{w_s}(\xi)$ of the slow turbulence component $w_s(t)$ given by Eq. (4.2),

$$\phi_{w_s}(\xi) = \sum_{j=0}^m a_j \xi^j, \quad 0 \leq \xi \leq \xi_H \quad (\text{H.1a})$$

$$= a_0 + a_1 \xi + \dots + a_m \xi^m, \quad (\text{H.1b})$$

where the extrapolation completes the description of $\phi_{w_s}(\xi)$ over the entire interval $0 \leq \xi \leq \infty$. We then integrate the resulting autocorrelation model to yield an approximation to the integral scale of the slow component $w_s(t)$; we then Fourier transform the resulting model to yield an analytical approximation to the power spectrum of the resulting model.

The extrapolation is carried out using the simple exponential decay model

$$\phi_{w_s}(\xi) = A e^{-\alpha \xi}, \quad \xi_H \leq \xi \leq \infty \quad (\text{H.2})$$

which completes the range of ξ not covered by the model of Eq. (H.1). Parameters A and α in Eq. (H.2) are set by requiring $\phi_{w_s}(\xi)$ and its first derivative to be continuous at the point of intersection $\xi = \xi_H$ of the two models. Using this method, the values a_0, a_1, \dots, a_m , and ξ_H *completely* describe the autocorrelation model of Eqs. (H.1) and (H.2).

Differentiating Eq. (H.1) gives

$$\begin{aligned}\phi'_{w_s}(\xi) &= a_1 + 2a_2\xi + \dots + ma_m\xi^{m-1} \\ &= \sum_{j=1}^m ja_j\xi^{j-1}, \quad 0 \leq \xi \leq \xi_H.\end{aligned}\quad (H.3)$$

Therefore, we have

$$\phi'_{w_s}(\xi_H) = \sum_{j=1}^m ja_j\xi_H^{j-1}, \quad (H.4)$$

where the prime denotes differentiation. Differentiating Eq. (H.2) gives

$$\phi'_{w_s}(\xi) = -\alpha Ae^{-\alpha\xi_H}, \quad \xi_H \leq \xi \leq \infty, \quad (H.5)$$

hence, we also have

$$\phi'_{w_s}(\xi_H) = -\alpha Ae^{-\alpha\xi_H}. \quad (H.6)$$

Dividing Eq. (H.5) by Eq. (H.2) gives

$$\frac{\phi'_{w_s}(\xi)}{\phi_{w_s}(\xi)} = -\alpha, \quad \xi_H \leq \xi \leq \infty. \quad (H.7)$$

Hence, if $\phi_{w_s}(\xi)$ is continuous at $\xi = \xi_H$, continuity of $\phi'_{w_s}(\xi)$ at $\xi = \xi_H$ requires from Eqs. (H.1), (H.3), and (H.7) that

$$\alpha = - \frac{\sum_{j=1}^m j a_j \xi_H^{j-1}}{\sum_{j=0}^m a_j \xi_H^j} , \quad (H.8)$$

which is an equation for α in terms of the parameters of the model of Eq. (H.1). To obtain an equation for A , we note from Eq. (H.2) that we must have

$$A = e^{\alpha \xi_H} \phi_{w_s}(\xi_H) \quad (H.9)$$

or,

$$A = e^{\alpha \xi_H} \sum_{j=0}^m a_j \xi_H^j , \quad (H.10)$$

where Eq. (H.10) follows from the continuity requirement of $\phi_{w_s}(\xi)$ at $\xi = \xi_H$ and Eq. (H.1). Equations (H.8) and (H.10) yield α and A from the parameters of the model, Eq. (H.1).

Expression for integral scale of slow component. To obtain an equation for the integral scale of the slow component, we require the integral $\int_0^\infty \phi_{w_s}(\xi) d\xi$. From Eqs. (H.1) and (H.2), we have

$$\begin{aligned} \int_0^\infty \phi_{w_s}(\xi) d\xi &= \int_0^{\xi_H} \left(\sum_{j=0}^m a_j \xi^j \right) d\xi + \int_{\xi_H}^\infty A e^{-\alpha \xi} d\xi \\ &= \sum_{j=0}^m a_j \frac{\xi^{j+1}}{j+1} \Big|_0^{\xi_H} + A \frac{e^{-\alpha \xi}}{-\alpha} \Big|_{\xi_H}^\infty \\ &= \sum_{j=0}^m a_j \frac{\xi_H^{j+1}}{j+1} + \frac{A}{\alpha} e^{-\alpha \xi_H} . \end{aligned} \quad (H.11)$$

The integral scale L is defined as

$$L \triangleq \frac{1}{\phi_w(0)} \int_0^\infty \phi_w(\xi) d\xi. \quad (\text{H.12})$$

From Eq. (H.11), we see that an expression for A/α is required, which from Eqs. (H.8) and (H.10) is

$$\frac{A}{\alpha} = - \frac{e^{\alpha \xi_H \left(\sum_{j=0}^m a_j \xi_H^j \right)^2}}{\sum_{j=1}^m j a_j \xi_H^{j-1}} \quad (\text{H.13})$$

Furthermore, from Eq. (H.1), we see that $\phi_{w_s}(0) = a_0$. Hence, applying the definition Eq. (H.12) to $\phi_{w_s}(\xi)$, we see from Eqs. (H.11) and (H.13) that

$$L_{w_s} = \frac{1}{a_0} \left[\sum_{j=0}^m \frac{a_j}{j+1} \xi_H^{j+1} - \frac{\left(\sum_{j=0}^m a_j \xi_H^j \right)^2}{\sum_{j=1}^m j a_j \xi_H^{j-1}} \right] \quad (\text{H.14})$$

which is an expression for the integral scale L_{w_s} of the slow turbulence component $w_s(t)$ in terms of the autocorrelation function parameters computed by the method described in Sec. 4.

Expression for power spectral density of slow component. By forming the Fourier transform of the above extrapolated autocorrelation function model, we obtain an expression for the power spectral density of the slow component of turbulence:

$$\begin{aligned} \Phi_{w_s}(k) &\triangleq \int_{-\infty}^{\infty} \phi_{w_s}(\xi) \cos(2\pi k \xi) d\xi \\ &= 2 \int_0^{\infty} \phi_{w_s}(\xi) \cos(2\pi k \xi) d\xi. \end{aligned} \quad (\text{H.15})$$

Introducing Eqs. (H.1) and (H.2) into Eq. (H.15) and carrying out the integration, we have

$$\Phi_{ws}(k) = 2 \sum_{j=0}^m a_j \int_0^{\xi_H} \xi^j \cos(2\pi k \xi) d\xi + 2A \int_{\xi_H}^{\infty} e^{-\alpha \xi} \cos(2\pi k \xi) d\xi. \quad (H.16)$$

Using formula 2.633-2 on p. 184 of Ref. 50 for the first of the above two integrals and a result in the well known Burington tables for the second integral, we obtain

$$\begin{aligned} \Phi_{ws}(k) = 2 \left\{ \sum_{j=0}^m a_j \left[\sum_{\ell=0}^j \ell! \binom{j}{\ell} \frac{\xi_H^{j-\ell}}{(2\pi k)^{\ell+1}} \sin(2\pi k \xi_H + \frac{1}{2} \ell \pi) \right. \right. \\ \left. \left. - j! \binom{j}{j} \frac{1}{(2\pi k)^{j+1}} \sin\left(\frac{1}{2} j \pi\right) \right] \right\} \\ - 2A \left\{ \frac{e^{-\alpha \xi_H}}{\alpha^2 + (2\pi k)^2} [2\pi k \sin(2\pi k \xi_H) - \alpha \cos(2\pi k \xi_H)] \right\}. \quad (H.17) \end{aligned}$$

However,

$$\ell! \binom{j}{\ell} = \ell! \frac{j!}{(j-\ell)! \ell!} = \frac{j!}{(j-\ell)!} \quad (H.18)$$

and

$$j! \binom{j}{j} = j!. \quad (H.19)$$

Furthermore, for any sum, we have

$$\sum_{j=0}^m \sum_{\ell=0}^j = \sum_{\ell=0}^m \sum_{j=\ell}^m. \quad (H.20)$$

Incorporating Eqs. (H.18) through (H.20) into Eq. (H.17) and slightly rearranging the result yields the desired expression for $\Phi_{ws}(k)$:

$$\begin{aligned}
\Phi_{w_s}(k) = & 2 \left\{ \sum_{\ell=0}^m \left[\sum_{j=\ell}^m a_j \frac{j!}{(j-\ell)!} \xi_H^{j+1} \right] \frac{\sin(2\pi\xi_H k + \frac{1}{2} \ell \pi)}{(2\pi\xi_H k)^{\ell+1}} \right\} \\
& - 2 \left[\sum_{j=0}^m a_j j! \frac{\sin\left(\frac{1}{2} j \pi\right)}{(2\pi k)^{j+1}} \right] \\
& - \frac{2Ae^{-\alpha\xi_H}}{\alpha^2 + (2\pi k)^2} [2\pi k \sin(2\pi\xi_H k) - \alpha \cos(2\pi\xi_H k)] , \quad (H.21)
\end{aligned}$$

where expressions for α and A are given by Eqs. (H.8) and (H.10). Equation (H.21) is a closed form expression for the power spectral density of the slow turbulence component $w_s(t)$ in terms of the autocorrelation function parameters determined by the method described in Sec. 4.

APPENDIX I

EVALUATION OF AN INTEGRAL

We evaluate below the integral defined by Eq. (5.84):

$$\gamma^{(3)} = \frac{1}{2} \left(\frac{a}{b} \right)^2 \int_{-\infty}^{\infty} \frac{[1+(2b+c)\xi^2]^2}{(1+b\xi^2)^2(1+c\xi^2)^2} d\xi. \quad (I.1)$$

To evaluate $\gamma^{(3)}$, we shall use the method described on pp. 584 to 587 of Ref. 52. Equation (I.1) is of the form of Eq. (119) on p. 584 of Ref. 52:

$$\int_{-\infty}^{\infty} \{\dots\} d\xi = 2\pi i \sum_k \text{Res}(a_k), \quad (I.2)$$

where the points a_k are the poles of the integrand that are located in the upper half plane. The poles of the integrand occur at the zeros of the denominator of Eq. (I.1), which are solutions to the equations

$$(1+b\xi^2)^2 = 0 \quad (1+c\xi^2)^2 = 0. \quad (I.3a,b)$$

These solutions are, respectively,

$$\xi_k = \pm i\sqrt{1/b} \quad \xi_k = \pm i\sqrt{1/c}. \quad (I.4a,b)$$

Therefore, the denominator of the integrand in Eq. (I.1) can be expressed as

$$(1+b\xi^2)^2(1+c\xi^2)^2 = b^2c^2 \left(\xi + \frac{i}{\sqrt{b}} \right)^2 \left(\xi - \frac{i}{\sqrt{b}} \right)^2 \left(\xi + \frac{i}{\sqrt{c}} \right)^2 \left(\xi - \frac{i}{\sqrt{c}} \right)^2. \quad (I.5)$$

The four factors in the right-hand side of Eq. (I.5) give rise to poles located at $\xi_k = -i/\sqrt{b}$, $\xi_k = +i/\sqrt{b}$, $\xi_k = -i/\sqrt{c}$, $\xi_k = +i/\sqrt{c}$ respectively. All four of these poles are of order *two*. Thus, there are two poles in the upper half plane which are located on the imaginary axis at $\xi_1 = i/\sqrt{b}$ and $\xi_2 = i/\sqrt{c}$

From p. 520 of Ref. 52, we see that for a pole of order m , we require the residues

$$\text{Res}(z_j) = \frac{1}{(m-1)!} \left[\frac{d^{m-1}}{dz^{m-1}} \left\{ (z-z_j)^m f(z) \right\} \right]_{z=z_j}, \quad (\text{I.6})$$

where for our application $m = 2$ and

$$f(z) = \frac{[1+(2b+c)z^2]^2}{b^2 c^2 \left(z^2 + \frac{1}{b}\right)^2 \left(z^2 + \frac{1}{c}\right)^2}. \quad (\text{I.7})$$

Thus, we have

$$(z-z_1)^2 f(z) = \frac{[1+(2b+c)z^2]^2}{b^2 c^2 \left(z + \frac{i}{\sqrt{b}}\right)^2 \left(z^2 + \frac{1}{c}\right)^2} \quad (\text{I.8})$$

and

$$(z-z_2)^2 f(z) = \frac{[1+(2b+c)z^2]^2}{b^2 c^2 \left(z^2 + \frac{1}{b}\right)^2 \left(z + \frac{i}{\sqrt{c}}\right)^2}. \quad (\text{I.9})$$

Differentiating Eqs. (I.8) and (I.9), as required by Eq. (I.6), and evaluating the resulting expressions at $z = z_1 = \frac{i}{\sqrt{b}}$ and $z = z_2 = \frac{i}{\sqrt{c}}$ respectively, we have after simplification

$$\text{Res}(z_1) = -i \frac{(b+c)(7b^2+c^2)}{4b^2} \left(\frac{\sqrt{b}}{c-b} \right)^3 \quad (\text{I.10})$$

and

$$\text{Res}(z_2) = i \frac{b(b^2+bc+2c^2)}{c^2} \left(\frac{\sqrt{c}}{c-b} \right)^3 . \quad (\text{I.11})$$

Therefore, from Eqs. (I.1) and (I.2), it follows that

$$\begin{aligned} \gamma^{(3)} &= \left(\frac{a}{b} \right)^2 \pi i [\text{Res}(z_1) + \text{Res}(z_2)] \\ &= \frac{\pi}{(c-b)^3} \left(\frac{a}{b} \right)^2 \left[\frac{(b+c)(7b^2+c^2)}{4b^{\frac{1}{2}}} - \frac{b(b^2+bc+2c^2)}{c^{\frac{1}{2}}} \right] , \quad (\text{I.12}) \end{aligned}$$

which is the result given by Eq. (5.85) in the main text of the report.

APPENDIX J

EVALUATION OF SECOND ORDER PARTIAL DERIVATIVES OF JOINT GAUSSIAN PROBABILITY DENSITY WITH RESPECT TO ITS PARAMETERS

The joint Gaussian probability density of Eq. (6.4) can be expressed as

$$p(y, \dot{y} | \sigma_y^2, \sigma_{\dot{y}}^2, \mu_{y\dot{y}}) \equiv p = \frac{1}{2\pi A^{1/2}} e^B \quad (J.1)$$

where

$$A \triangleq \sigma_y^2 \sigma_{\dot{y}}^2 - \mu_{y\dot{y}}^2 \quad (J.2)$$

and

$$B \triangleq \frac{N}{2A} = \frac{-\sigma_{\dot{y}}^2 y^2 + 2\mu_{y\dot{y}} y\dot{y} - \sigma_y^2 \dot{y}^2}{2(\sigma_y^2 \sigma_{\dot{y}}^2 - \mu_{y\dot{y}}^2)}, \quad (J.3)$$

where

$$N \triangleq -\sigma_{\dot{y}}^2 y^2 + 2\mu_{y\dot{y}} y\dot{y} - \sigma_y^2 \dot{y}^2 \quad (J.4)$$

Denoting derivatives by superscripts in parentheses, we have by differentiating Eq. (J.1),

$$p^{(1)} = p \left(B^{(1)} - \frac{A^{(1)}}{2A} \right) \quad (J.5)$$

and

$$p^{(2)} = p \left\{ -\frac{1}{A} \left[\frac{A^{(2)}}{2} - \frac{3(A^{(1)})^2}{4A} + A^{(1)} B^{(1)} \right] + B^{(2)} + (B^{(1)})^2 \right\}. \quad (J.6)$$

By differentiating the left-hand portion of Eq. (J.3), we have

$$B^{(1)} = \frac{N}{2A} \left(\frac{N^{(1)}}{N} - \frac{A^{(1)}}{A} \right) \quad (J.7)$$

and

$$B^{(2)} = \frac{N}{2A} \left[\frac{N^{(2)}}{N} - \frac{2N^{(1)}A^{(1)}}{NA} + 2 \left(\frac{A^{(1)}}{A} \right)^2 - \frac{A^{(2)}}{A} \right] \quad (J.8)$$

Finally, combining Eqs. (J.7) and Eq. (J.8) with Eq. (J.6), we obtain an expression for the second derivative of $p(y, \dot{y} | \sigma_y^2, \sigma_{\dot{y}}^2, \mu_{y\dot{y}})$ in terms of the derivatives of A and N taken with respect to the same variable:

$$p^{(2)} = \frac{p}{2A^2} \left\{ \frac{3}{2} (A^{(1)})^2 - A^{(2)} A + AN^{(2)} - 3A^{(1)} N^{(1)} \right. \\ \left. + \left[\frac{3(A^{(1)})^2}{A} - A^{(2)} \right] N + \frac{(N^{(1)})^2}{2} - \frac{A^{(1)}}{A} N^{(1)} N + \frac{(A^{(1)})^2}{2A^2} N^2 \right\}. \quad (J.9)$$

Equation (J.9) is a general expression that can be used to evaluate $p^{(2,0,0)}$, $p^{(0,2,0)}$, and $p^{(0,0,2)}$ for use in Eq. (6.10) by differentiating A and N with respect to the appropriate variables σ_y , $\sigma_{\dot{y}}$, or $\mu_{y\dot{y}}$ as indicated by the superscript notation defined by Eq. (6.11). A and N are defined by Eqs. (J.2) and (J.4).

Let us turn now to evaluation of the cross-partial derivatives required for the last three terms in Eq. (6.10). These terms are $p^{(1,1,0)}$, $p^{(1,0,1)}$, and $p^{(0,1,1)}$. In this case, we shall use a double superscript notation to denote partial derivatives with respect to whatever *two* variables are required to evaluate these terms.

We can immediately write from Eq. (J.5)

$$p^{(1,0)} = p \left(B^{(1,0)} - \frac{A^{(1,0)}}{2A} \right), \quad (J.10)$$

and differentiating this expression with respect to the second variable, we find

$$\begin{aligned} p^{(1,1)} = p \left\{ -\frac{1}{A} \left[\frac{A^{(1,1)}}{2} - \frac{3A^{(1,0)}A^{(0,1)}}{4A} \right. \right. \\ \left. \left. + \frac{1}{2} \left(A^{(1,0)}B^{(0,1)} + A^{(0,1)}B^{(1,0)} \right) \right] \right. \\ \left. + B^{(1,0)}B^{(0,1)} + B^{(1,1)} \right\}. \end{aligned} \quad (J.11)$$

From Eq. (J.7), we can immediately write

$$B^{(1,0)} = \frac{N}{2A} \left(\frac{N^{(1,0)}}{N} - \frac{A^{(1,0)}}{A} \right) \quad (J.12)$$

and

$$B^{(0,1)} = \frac{N}{2A} \left(\frac{N^{(0,1)}}{N} - \frac{A^{(0,1)}}{A} \right). \quad (J.13)$$

Differentiating Eq. (J.12) with respect to the second variable gives us $B^{(1,1)}$:

$$\begin{aligned} B^{(1,1)} = \frac{1}{2A} \left\{ N^{(1,1)} - \frac{A^{(1,0)}N^{(0,1)}}{A} - \frac{A^{(0,1)}N^{(1,0)}}{A} \right. \\ \left. + \frac{2A^{(1,0)}A^{(0,1)}N}{A^2} - \frac{A^{(1,1)}N}{A} \right\}. \end{aligned} \quad (J.14)$$

Finally, substituting Eqs. (J.12) to (J.14) into Eq. (J.11) yields the desired expression for $p^{(1,1)}$:

$$\begin{aligned}
 p^{(1,1)} = \frac{p}{2A^2} \left\{ \frac{3}{2} A^{(1,0)} A^{(0,1)} - A^{(1,1)} A + N^{(1,1)} - \frac{3}{2} A^{(0,1)} N^{(1,0)} \right. \\
 - \frac{3}{2} A^{(1,0)} N^{(0,1)} + \left[\frac{3A^{(1,0)} A^{(0,1)}}{A} - A^{(1,1)} \right] N \frac{N^{(1,0)} N^{(0,1)}}{2} \\
 \left. - \frac{A^{(0,1)}}{2A} N^{(1,0)} N - \frac{A^{(1,0)}}{2A} N^{(0,1)} N + \frac{A^{(1,0)} A^{(0,1)}}{2A^2} N^2 \right\}.
 \end{aligned}
 \tag{J.15}$$

Equation (J.15) is a general expression that can be used to evaluate $p^{(1,1,0)}$, $p^{(1,0,1)}$, and $p^{(0,1,1)}$ for use in Eq. (6.10). In evaluating the various terms in Eq. (J.15), the double superscript notation is used to denote derivatives with respect to whatever *two* variables in $p^{(1,1,0)}$, $p^{(1,0,1)}$, and $p^{(0,1,1)}$ the partial derivatives are taken with respect to. A and N are defined by Eqs. (J.2) and (J.4).

As a check of Eqs. (J.9) and (J.15), we note that the expression for $p^{(1,1)}$ should reduce to the expression for $p^{(2)}$ when we substitute the right-hand sides of the following expressions into Eq. (J.15): $A^{(1,0)} = A^{(1)}$, $A^{(0,1)} = A^{(1)}$, $A^{(1,1)} = A^{(2)}$, $N^{(1,0)} = N^{(1)}$, $N^{(0,1)} = N^{(1)}$, $N^{(1,1)} = N^{(2)}$. Carrying out these substitutions reduces Eq. (J.15) to Eq. (J.9).

We shall now use Eq. (J.9) to evaluate the terms $p^{(2,0,0)}$, $p^{(0,2,0)}$, and $p^{(0,0,2)}$ for use in Eq. (6.10), and following that we shall use Eq. (J.15) to evaluate $p^{(1,1,0)}$, $p^{(1,0,1)}$, and $p^{(0,1,1)}$.

To evaluate $p^{(2,0,0)}(y, \dot{y} | \sigma_y^2, \sigma_{\dot{y}}^2, \mu_{y\dot{y}})$, we identify all derivatives in Eq. (J.9) as derivatives with respect to σ_y^2 [as indicated by Eq. (6.11)]. Therefore, from Eqs. (J.2) and (J.4) we have for evaluation of $p^{(2,0,0)}(y, \dot{y} | \sigma_y^2, \sigma_{\dot{y}}^2, \mu_{y\dot{y}})$:

$$A^{(1)} = \sigma_{\dot{y}}^2, \quad A^{(2)} = 0 \quad (J.16)$$

$$N^{(1)} = -\dot{y}^2, \quad N^{(2)} = 0. \quad (J.17)$$

When Eqs. (J.1) to (J.4) and Eqs. (J.16) and (J.17) are substituted into Eq. (J.9) we obtain the desired expression for $p^{(2,0,0)}(y, \dot{y} | \sigma_y^2, \sigma_{\dot{y}}^2, \mu_{y\dot{y}})$.

In like manner, we have developed the following table which gives the evaluates of $A^{(1)}$, $A^{(2)}$, $N^{(1)}$, and $N^{(2)}$ required for use in Eq. (J.9) to obtain expressions for $p^{(2,0,0)}$, $p^{(0,2,0)}$, and $p^{(0,0,2)}$.

	$A^{(1)}$	$A^{(2)}$	$N^{(1)}$	$N^{(2)}$
$p^{(2,0,0)}$	$\sigma_{\dot{y}}^2$	0	$-\dot{y}^2$	0
$p^{(0,2,0)}$	σ_y^2	0	$-y^2$	0
$p^{(0,0,2)}$	$-2\mu_{y\dot{y}}$	-2	$2y\dot{y}$	0

Table J.1. Evaluates of $A^{(1)}$, $A^{(2)}$, $N^{(1)}$, and $N^{(2)}$ for substitution into Eq. (J.9) to determine expressions for $p^{(2,0,0)}$, $p^{(0,2,0)}$, and $p^{(0,0,2)}$. Quantities p , A , B , and N are given by Eqs. (J.1) to (J.4) respectively.

Similarly, we obtain Table J.2 for evaluation of $p^{(1,1,0)}$, $p^{(1,0,1)}$, and $p^{(0,1,1)}$ by Eq. (J.15):

	$A(1,0)$	$A(0,1)$	$A(1,1)$	$N(1,0)$	$N(0,1)$	$N(1,1)$
$p(1,1,0)$	$\sigma_{\dot{y}}^2$	σ_y^2	1	$-\dot{y}^2$	$-y^2$	0
$p(1,0,1)$	$\sigma_{\dot{y}}^2$	$-2\mu_{y\dot{y}}$	0	$-\dot{y}^2$	$2y\dot{y}$	0
$p(0,1,1)$	σ_y^2	$-2\mu_{y\dot{y}}$	0	$-y^2$	$2y\dot{y}$	0

Table J.2. Evaluates of $A(1,0)$, $A(0,1)$, $A(1,1)$, $N(1,0)$, $N(0,1)$, and $N(1,1)$ for substitution into Eq. (J.15) to determine expressions for $p(1,1,0)$, $p(1,0,1)$, and $p(0,1,1)$. Quantities p , A , B , and N are given by Eqs. (J.1) to (J.4) respectively.

When Eqs. (J.2), (J.4), and the parameter values given in Table J.1 are substituted into Eq. (J.9), we obtain the expressions for $p(2,0,0)$, $p(0,2,0)$, and $p(0,0,2)$ given by Eqs. (6.15) to (6.17) respectively, and when Eqs. (J.2), (J.4), and the parameter values given in Table J.2 are substituted into Eq. (J.15), we obtain the expressions for $p(1,1,0)$, $p(1,0,1)$, and $p(0,1,1)$ given by Eqs. (6.18) to (6.20) respectively. Since these derivatives are shown evaluated at the expected values of the parameters $\sigma_{\dot{y}}^2$, σ_y^2 , and $\mu_{y\dot{y}}$, bars are shown over these parameters in Eqs. (6.15) to (6.20); furthermore, we have used the fact shown by Eq. (6.74) that

$$\overline{\mu_{y\dot{y}}} \equiv E_{\sigma_f} \{\mu_{y\dot{y}}\} = 0 \quad (J.18)$$

in the expressions of Eqs. (6.15) to (6.20).

APPENDIX K

DERIVATION OF INPUT-RESPONSE RELATIONS FOR INSTANTANEOUS CROSS-SPECTRAL DENSITIES OF NONSTATIONARY STOCHASTIC PROCESSES

Here, we derive input-response relations for instantaneous cross-spectral densities that are direct extensions of the results derived in Ref. 34. Let us define the instantaneous cross-correlation function of two real, generally nonstationary stochastic processes $\{x_j(t)\}$ and $\{x_k(t)\}$ as

$$\phi_{x_j x_k}(\tau, t) \triangleq E\{x_j(t - \frac{\tau}{2})x_k(t + \frac{\tau}{2})\}. \quad (K.1)$$

When the two processes are identical - i.e., when $j = k$, and therefore $x_j(t) = x_k(t) = \phi_{x_j x_j}(\tau, t)$ is an even function of τ as is immediately apparent from Eq. (K.1). However, when $x_j(t) \neq x_k(t)$, $\phi_{x_j x_k}(\tau, t)$ is not generally an even function of τ . The definition, Eq. (K.1), is a direct extension of the definition, Eq. (7), of Ref. 34.

We define the instantaneous cross-spectral density $\Phi_{x_j x_k}(f, t)$ of the two processes $\{x_j(t)\}$ and $\{x_k(t)\}$ as the Fourier transform with respect to τ of $\phi_{x_j x_k}(\tau, t)$ - i.e.,

$$\Phi_{x_j x_k}(f, t) \triangleq \int_{-\infty}^{\infty} \phi_{x_j x_k}(\tau, t) e^{-i2\pi f\tau} d\tau, \quad (K.2)$$

which is a direct extension of the definition, Eq. (9a), of Ref. 34. However, in the present case $\Phi_{x_j x_k}(f, t)$ is not generally real and an even function of f as it is in the case where $x_j(t) = x_k(t)$. Let us further define the Fourier transform with respect to t of $\Phi_{x_j x_k}(f, t)$ as

$$\tilde{\Phi}_{x_j x_k}(f, \nu) \triangleq \int_{-\infty}^{\infty} \Phi_{x_j x_k}(f, t) e^{i2\pi \nu t} dt \quad (K.3)$$

which is a direct extension of Eq. (27) of Ref. 34.

We may relate $\Phi_{x_j x_k}(f, \nu)$ directly to the Fourier transforms of the sample functions $x_j(t)$ and $x_k(t)$. Substituting Eqs. (K.1) and (K.2) into Eq. (K.3) and interchanging the orders of expectation and integration, we have

$$\tilde{\Phi}_{x_j x_k}(f, \nu) = E \left\{ \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} x_j(t - \frac{\tau}{2}) x_k(t + \frac{\tau}{2}) e^{-i2\pi(f\tau - \nu t)} d\tau dt \right\}. \quad (K.4)$$

Let us now transform to the new variables of integration

$$t_1 = t - \frac{\tau}{2}, \quad t_2 = t + \frac{\tau}{2}; \quad (K.5)$$

hence,

$$\tau = t_2 - t_1, \quad t = \frac{t_1 + t_2}{2}, \quad (K.6)$$

where $|\partial(\tau, t)/\partial(t_1, t_2)| = 1$. Substitution of Eqs. (K.5) and (K.6) into Eq. (K.4) and using

$$d\tau dt = |\partial(\tau, t)/\partial(t_1, t_2)| dt_1 dt_2 = dt_1 dt_2, \quad (K.7)$$

we have after minor rearrangements

$$\begin{aligned} \tilde{\Phi}_{x_j x_k}(f, \nu) &= E \left\{ \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} x_j(t_1) x_k(t_2) e^{-i2\pi[(f - \frac{\nu}{2})t_2 - (f + \frac{\nu}{2})t_1]} dt_1 dt_2 \right\} \\ &= E \left\{ \int_{-\infty}^{\infty} x_j(t_1) e^{i2\pi(f + \frac{\nu}{2})t_1} dt_1 \int_{-\infty}^{\infty} x_k(t_2) e^{-i2\pi(f - \frac{\nu}{2})t_2} dt_2 \right\} \\ &= E \left\{ X_j^*(f + \frac{\nu}{2}) X_k(f - \frac{\nu}{2}) \right\}, \end{aligned} \quad (K.8)$$

where we have defined

$$X_j(f) \triangleq \int_{-\infty}^{\infty} x_j(t) e^{-i2\pi ft} dt \quad (K.9a)$$

$$X_k(f) = \int_{-\infty}^{\infty} x_k(t) e^{-i2\pi ft} dt, \quad (K.9b)$$

and where the superscript asterisk in Eq. (K.8) denotes the complex conjugate. Thus, from Eq. (K.8) we see that $\tilde{\Phi}_{x_j x_k}(f, \nu)$ can be expressed directly in terms of an expectation of the cross-products of the Fourier transforms of the sample functions of the two processes $\{x_j(t)\}$ and $\{x_k(t)\}$.

The Fourier mate to Eq. (K.2) is

$$\phi_{x_j x_k}(\tau, t) = \int_{-\infty}^{\infty} \Phi_{x_j x_k}(f, t) e^{i2\pi f \tau} df. \quad (K.10)$$

Combining the evaluations of Eqs. (K.1) and (K.10) at $\tau = 0$ gives

$$\phi_{x_j x_k}(0, t) \equiv E\{x_j(t)x_k(t)\} = \int_{-\infty}^{\infty} \Phi_{x_j x_k}(f, t) df, \quad (K.11)$$

which is the extension of Eq. (12a) of Ref. 34 to instantaneous cross-spectral densities. For any time t , integration of the (complex) instantaneous cross-spectral density $\Phi_{x_j x_k}(f, t)$ over all f gives the expected value of the product $x_j(t)x_k(t)$ at that same instant of time t .

Cross-spectral density input-response relations. Consider the response $\{y(t)\}$ of a linear time-invariant system to an input process $\{x(t)\}$. Let $h(t)$ denote the unit impulse response of the system. For any input sample function $x(t)$ we have

$$y(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau, \quad (K.12)$$

as in Eq. (1.10b). Let $H(f)$ denote the complex frequency response of the system as defined by Eq. (1.9), and let $X(f)$ and $Y(f)$ denote the Fourier transforms of $x(t)$ and $y(t)$ defined in the same manner as in Eq. (1.9). Then, it is well known [e.g., p. 57 of Ref. 29] that $X(f)$, $Y(f)$, and $H(f)$ are related by the product

$$Y(f) = X(f)H(f) . \quad (K.13)$$

Let $\{x_j(t)\}$ and $\{x_k(t)\}$ denote two different input processes and let $\{y_j(t)\}$ and $\{y_k(t)\}$ denote the corresponding response processes. That is, each sample function from process $\{x_j(t)\}$ generates a response sample function by the relation

$$y_j(t) = \int_{-\infty}^{\infty} h_j(\tau)x_j(t-\tau)d\tau , \quad (K.14)$$

and each sample function from $\{x_k(t)\}$ also generates a comparable response sample function

$$y_k(t) = \int_{-\infty}^{\infty} h_k(\tau)x_k(t-\tau)d\tau . \quad (K.15)$$

Impulse response functions $h_j(t)$ and $h_k(t)$ are potentially different. The frequency domain counterparts of these input-response relationships are

$$Y_j(f) = X_j(f)H_j(f) \quad (K.16)$$

and

$$Y_k(f) = X_k(f)H_k(f) . \quad (K.17)$$

From Eqs. (K.16) and (K.17), we therefore have

$$E\{Y_j^*(f + \frac{\nu}{2})Y_k(f - \frac{\nu}{2})\} = E\{X_j^*(f + \frac{\nu}{2})X_k(f - \frac{\nu}{2})\}H_j^*(f + \frac{\nu}{2})H_k(f - \frac{\nu}{2}), \quad (K.18)$$

or, applying the definition, Eq. (K.8), to input processes, output processes, and system complex frequency response functions, we have

$$\tilde{\Phi}_{y_j y_k}(f, \nu) = \tilde{\Phi}_{x_j x_k}(f, \nu) \tilde{\Phi}_{h_j h_k}(f, \nu), \quad (K.19)$$

where no expectation operation is required in defining $\tilde{\Phi}_{h_j h_k}(f, \nu)$ because $h_j(t)$, $h_k(t)$ and their transforms are here assumed to be deterministic.

Equation (K.19) expresses the transformed instantaneous cross-spectral density of the response processes $\{y_j(t)\}$ and $\{y_k(t)\}$ as the product of the transformed instantaneous cross-spectra of the input processes and system impulse response functions. Hence, from the Fourier mate to Eq. (K.3),

$$\Phi_{x_j x_k}(f, t) = \int_{-\infty}^{\infty} \tilde{\Phi}_{x_j x_k}(f, \nu) e^{-i2\pi \nu t} d\nu, \quad (K.20)$$

and the analogous relations for the response cross-spectra system impulse response cross-spectra, we have by applying the convolution theorem to Eq. (K.19):

$$\Phi_{y_j y_k}(f, t) = \int_{-\infty}^{\infty} \Phi_{x_j x_k}(f, t-u) \Phi_{h_j h_k}(f, u) du. \quad (K.21)$$

Equations (K.19) and (K.21) are the instantaneous cross-spectral density input-response relations that are direct extensions of the instantaneous auto-spectral density input-response relations, Eqs. (39) and (40) of Ref. 34.

Reduction to the case of stationary input processes. Let us now consider the case where $\Phi_{x_j x_k}(\tau, t)$ defined by Eq. (K.1)

is independent of t - i.e., is dependent only on the time difference τ . In this case, we see from Eq. (K.2) that $\phi_{x_j x_k}(f, t)$

also is independent of t . Denote this independence by replacing t by a vertically centered dot. It then follows immediately from Eq. (K.21) that $\phi_{y_j y_k}(f, t)$ also is independent of t , which

we also shall denote by a vertically centered dot. Thus, when $\phi_{x_j x_k}(\tau, t)$ is independent of t , Eq. (K.21) reduces to

$$\phi_{y_j y_k}(f, \cdot) = \phi_{x_j x_k}(f, \cdot) \int_{-\infty}^{\infty} \phi_{h_j h_k}(f, u) du . \quad (K.22)$$

However, from the counterpart of Eq. (K.3) applied to $h_j h_k$ - i.e.,

$$\tilde{\phi}_{h_j h_k}(f, \nu) \triangleq \int_{-\infty}^{\infty} \phi_{h_j h_k}(f, t) e^{i2\pi \nu t} dt , \quad (K.23)$$

we have by setting $\nu = 0$ in Eq. (K.23),

$$\begin{aligned} \int_{-\infty}^{\infty} \phi_{h_j h_k}(f, t) dt &= \tilde{\phi}_{h_j h_k}(f, 0) \\ &= H_j^*(f) H_k(f) , \end{aligned} \quad (K.24)$$

where the second line follows directly from Eq. (K.8) applied to $h_j h_k$ rather than $x_j x_k$. Combining Eqs. (K.22) and (K.24) yields

$$\phi_{y_j y_k}(f, \cdot) = \phi_{x_j x_k}(f, \cdot) H_j^*(f) H_k(f) . \quad (K.25)$$

Finally, by applying Eq. (K.11) to the response $y_j y_k$, we have from Eqs. (K.25) for stationary input processes,

$$E\{y_j(t)y_k(t)\} = \int_{-\infty}^{\infty} \Phi_{x_j x_k}(f, \cdot) H_j^*(f) H_k(f) df , \quad (K.26)$$

which is valid whenever the instantaneous input cross-correlation function, Eq. (K.1), is independent of t .

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16. Abstract A nonGaussian three component model of atmospheric turbulence is postulated that accounts for readily observable features of turbulence velocity records, their autocorrelation functions, and their spectra. Methods for computing probability density functions and mean exceedance rates of a generic aircraft response variable are developed using nonGaussian turbulence characterizations readily extracted from velocity recordings. A maximum likelihood method is developed for optimal estimation of the integral scale and intensity of records possessing von Karman transverse or longitudinal spectra. Formulas for the variances of such parameter estimates are developed. The maximum likelihood and least-square approaches are combined to yield a method for estimating the autocorrelation function parameters of a two component model of turbulence.			
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